ANALYTIC GEOMETRY

ZWET AND HOPKINS

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ANALYTIC GEOMETRY

A SERIES OF MATHEMATICAL TEXTS

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ANALYTIC GEOMETRY

AND

PRINCIPLES OF ALGEBRA

BY

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PREFACE

The present work combines with analytic geometry a number of topics traditionally treated in college algebra that depend upon or are closely associated with geometric representation. Through this combination it becomes possible to show the student more directly the meaning and the usefulness of these subjects.

The idea of coordinates is so simple that it might (and perhaps should) be explained at the very beginning of the study of algebra and geometry. Real analytic geometry, however, begins only when the equation in two variables is interpreted as defining a locus. This idea must be introduced very gradually, as it is difficult for the beginner to grasp. The familiar loci, straight line and circle, are therefore treated at great length.

Simultaneous linear equations present themselves naturally in connection with the intersection of straight lines and lead to an early introduction of determinants, whose broad usefulness is most apparent in analytic geometry.

The study of the circle calls for a discussion of quadratic equations which again leads to complex numbers. The geometric representation of complex numbers will present no great difficulty because the student is now somewhat familiar with the idea of variables, of coordinates, and even vectors (in a plane).

The discussion of the conic sections is preceded by the study, especially the plotting, of curves of the form $y = f(x)$, where $f(x)$ is a polynomial of the second, third, etc. degree. In connection with this the solution of numerical algebraic equations can be given a geometric setting.

In the chapters on the conic sections only the most essential properties of these curves are given in the text; thus, poles and polars are discussed only in connection with the circle.

Great care has been taken in presenting the fundamental problem of finding the slope of a curve. It seemed desirable and quite feasible to introduce the idea of the derivative (of a polynomial only) in connection with the discussion of algebraic equations. The calculus method of finding the slope of a conic section has therefore been explained, in addition to the direct geometric method.

The treatment of solid analytic geometry follows more the usual lines. But, in view of the application to mechanics, the idea of the vector is given some prominence; and the representation of a function of two variables by contour lines as well as by a surface in space is explained and illustrated by practical examples.

The exercises have been selected with great care in order not only to furnish sufficient material for practice in algebraic work but also to stimulate independent thinking and to point out the applications of the theory to concrete problems. The number of exercises is sufficient to allow the instructor to make a choice.

To reduce the course presented in this book to about one half its extent, the parts of the text in small type, the chapters on solid analytic geometry, and the more difficult problems throughout may be omitted.

ALEXANDER ZIWET,
L. A. HOPKINS,
E. R. HEDRICK, EDITOR.

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ANALYTIC GEOMETRY

PLANE ANALYTIC GEOMETRY

CHAPTER I

COORDINATES

1. Location of a Point on a Line. The position of a point P (Fig. 1) on a line is fully determined by its distance OP from a fixed point O on the line, if we know on which *side* of O the point P is situated (to the right or to the left of O in Fig. 1). Let us agree, for instance, to count distances to the right of O as positive, and distances to the left of O as negative; this is indicated in Fig. 1 by the arrowhead which marks the *positive sense* of the line.

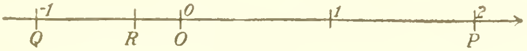

FIG. 1

The fixed point O is called the ***origin.*** The distance OP, taken with the sign $+$ if P lies, let us say, on the right, and with the sign $-$ when P lies on the opposite side, is called the ***abscissa*** of P.

It is assumed that the unit in which the distances are measured (inches, feet, miles, etc.) is known. On a geographical map, or on a plan of a lot or building, this unit is indicated by the scale. In Fig. 1, the unit of measure is one inch, the abscissa of P is $+2$, that of Q is -1, that of R is $-1/3$.

2. Determination of a Point by its Abscissa. Let us select, on a given line, an arbitrary *origin* O, a *unit of measure*, and a definite *sense* as positive. Then any real number, such as 5, -3, 7.35, $-\sqrt{2}$, regarded as the *abscissa* of a point P, fully determines the position of P on the line. Conversely, every point on the line has one and only one abscissa.

The abscissa of a point is usually denoted by the letter x, which, in analytic geometry as in algebra, may represent any real or complex number.

To represent a *real* point the abscissa must be a real number. If in any problem the abscissa x of a point is not a real number, there exists no real point satisfying the conditions of the problem.

EXERCISES

1. What is the abscissa of the origin?

2. With the inch as unit of length, mark on a line the points whose abscissas are: 3, -2, $\sqrt{3}$, -1.25, $-\sqrt{5}$, $\frac{2}{3}$, $-\frac{1}{6}$.

3. On a railroad line running east and west, if the station B is 20 miles east of the station A and the station C is 33 miles east of A, what are the abscissas of A and C for B as origin, the sense eastward being taken as positive?

4. On a Fahrenheit thermometer, what is the positive sense? What is the unit of measure? What is the meaning of the reading 65°? What is meant by $-7°$?

5. A water gauge is a vertical post carrying a scale; the *mean* water level is generally taken as origin. If the water stands at $+7$ on one day and at -11 the next day, the unit being the inch, how much has the water fallen?

6. If x_1, x_2 (read: x one, x two) are the abscissas of any two points P_1, P_2 on a given line, show that the abscissa of the midpoint between P_1 and P_2 is $\frac{1}{2}(x_1 + x_2)$. Consider separately the cases when P_1, P_2 lie on the same side of the origin O and when they lie on opposite sides.

3. Ratio of Division. A segment AB (Fig. 2) of a straight line being given, it is shown in elementary geometry how to find the point C that divides AB in a given ratio k. Thus, if $k = \frac{2}{5}$, the point C such that

$$\frac{AC}{AB} = \frac{2}{5}$$

is found as follows. On any line through A lay off $AD = 2$ and $AE = 5$; join B and E. Then the parallel to BE through D meets AB at the required point C.

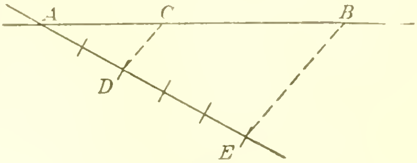

FIG. 2

Analytically, the problem of dividing a line in a given ratio is solved as follows. On the line AB (Fig. 3) we choose a point O as origin and assign a positive sense. Then the abscissas x_1 of A and x_2 of B are known. To find a point C

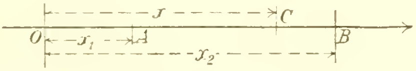

FIG. 3

which divides AB in the *ratio of division* $k = AC/AB$, let us denote the unknown abscissa of C by x. Then we have

$$AC = x - x_1, \quad AB = x_2 - x_1;$$

hence the abscissa x of C must satisfy the condition

$$\frac{x - x_1}{x_2 - x_1} = k,$$

whence

$$x = x_1 + k(x_2 - x_1);$$

or, if we write Δx (read: delta x) for the "difference of the x's," *i.e.* $\Delta x = x_2 - x_1$,

$$x = x_1 + k \cdot \Delta x.$$

Thus, if the abscissas of A and B are 2 and 7, the abscissas of the points that divide AB in the ratios $\frac{1}{5}$, $\frac{1}{2}$, $\frac{6}{5}$, $\frac{3}{2}$ are 3, $4\frac{1}{2}$, 8, $9\frac{1}{2}$, respectively. Check these results by geometric construction.

If the segments AC and AB have the same sense, the division ratio k is positive. For example, in Fig. 3, the point C lies between A and B; hence the division ratio k is a positive proper fraction. If the division ratio k is negative, the segments AC and AB must have opposite sense, so that B and C lie on the opposite sides of A.

If the abscissas of A and B are again 2 and 7, the abscissa x of C when $k = 2, -1, -\frac{2}{5}, -.2$ will be $12, -3, 0, 1$, respectively. Illustrate this by a figure, and check by the geometric construction.

4. Location of a Point in a Plane. To locate a point in a plane, that is, to determine its position in a plane, we may proceed as follows. Draw two lines at right angles in the plane; on each of these take the point of intersection O as origin, and assign a definite positive sense to each line, *e.g.* by marking each line with an arrowhead. It is usual to mark the positive sense of one line by affixing the letter x to it, and the positive sense of the other line by affixing the letter y to it, as in Fig. 4. These two lines are then called the ***axes of coordinates***, or simply the ***axes***. We distinguish them by calling the line Ox the x-axis, or axis of abscissas, and the line Oy the y-axis, or axis of ordinates. Now project the point P on each axis, *i.e.* let fall the perpendiculars PQ, PR from P on the axes. The point Q has the abscissa $OQ = x$ on the axis Ox. The point R has the abscissa $OR = y$ on the axis Oy. The distance $OQ = RP = x$ is called the ***abscissa*** of P, and

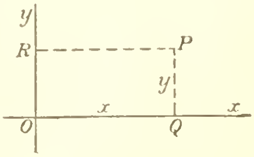

FIG. 4

$OR = QP = y$ is called the ***ordinate*** of P. The position of the point P in the plane is fully determined if its abscissa x and its ordinate y are both given. The two numbers x, y are also called the ***coordinates*** of the point P.

5. Signs of the Coordinates. Quadrants. It is clear from Fig. 4 that x and y are the perpendicular distances of the point P from the two axes. It should be observed that each of these numbers may be positive or negative, as in § 1.

The two axes divide the plane into four compartments distinguished as in trigonometry as the first, second, third, and fourth ***quadrants*** (Fig. 5). It is readily seen that any point in the first quadrant has both its coordinates positive. What are the signs of the coordinates in the other quadrants? What are the coordinates of the origin O? What are the coordinates of a point on one of the axes? It is customary to name the abscissa first and then the ordinate; thus the point $(-3, 5)$ means the point whose abscissa is -3 and whose ordinate is 5.

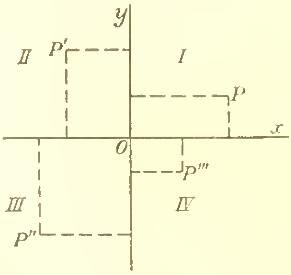

FIG. 5

Every point in the plane has two definite real numbers as coordinates; conversely, *to every pair of real numbers corresponds one and only one point of the plane.*

Locate the points: $(6, -2)$, $(0, 7)$, $(2-\sqrt{3}, \frac{2}{3})$, $(-4, 2\sqrt{2})$, $(-5, 0)$.

6. Units. It may sometimes be convenient to choose the unit of measure for the abscissa of a point different from the unit of measure for the ordinate. Thus, if the same unit, say one inch, were taken for abscissa and ordinate, the point $(3, 48)$ might fall beyond the limits of the paper. To avoid this we may lay off the ordinate on a scale of $\frac{1}{8}$ inch. When different units are used, the unit used on each axis should always be indicated in the drawing. When nothing is said to the contrary, the units for abscissas and ordinates are always understood to be the same.

7. Oblique Axes. The position of a point in a plane can also be determined with reference to two axes that are *not* at right angles; but the angle ω between these axes must be given (Fig. 6). The abscissa and the ordinate of the point P are then the segments $OQ = x$, $OR = y$ cut off on the axes by the parallels through P to the axes. If $\omega = \frac{1}{2}\pi$, *i.e.* if the axes are at right angles, we have the case of ***rectangular coordinates*** discussed in §§ 4, 5. In what follows, the axes are always taken at right angles unless the contrary is definitely stated.

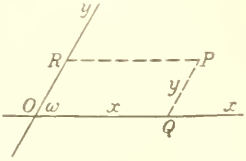

FIG. 6

8. Distance of a Point from the Origin. For the distance $r = OP$ (Fig. 7) of the point P from the origin O we have from the right-angled triangle OQP:

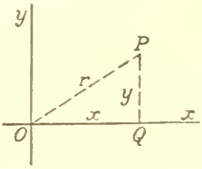

FIG. 7

$$r = \sqrt{x^2 + y^2},$$

where x, y are the coordinates of P.

If the axes are oblique (Fig. 8), with the angle $xOy = \omega$, we have, from the triangle OQP, in which the angle at Q is equal to $\pi - \omega$,* by the cosine law of trigonometry,

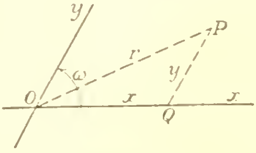

FIG. 8

$$r = \sqrt{x^2 + y^2 - 2\,xy\cos(\pi - \omega)} = \sqrt{x^2 + y^2 + 2\,xy\cos\omega}.$$

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* In advanced mathematics, angles are generally measured in radians, the symbol π denoting an angle of 180°.

Notice that these formulas hold not only when the point P lies in the first quadrant, but quite generally wherever the point P may be situated. Draw the figures for several cases.

9. Distance between Two Points. By Fig. 9, the distance $d = P_1P_2$ between two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ can be found if the coordinates of the two points are given. For in the triangle P_1QP_2 we have

$$P_1Q = x_2 - x_1, \quad QP_2 = y_2 - y_1;$$

hence

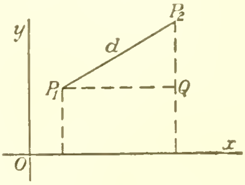


FIG. 9

$$(1) \qquad d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

If we write Δx (§ 3) for the "difference of the x's" and Δy for the "difference of the y's", *i.e.*

$$\Delta x = x_2 - x_1 \quad \text{and} \quad \Delta y = y_2 - y_1,$$

the formula for the distance has the simple form

$$(2) \qquad d = \sqrt{(\Delta x)^2 + (\Delta y)^2};$$

or, in words,

The distance between any two points is equal to the square root of the sum of the squares of the differences between their corresponding coordinates.

Draw the figure showing the distance between two points (like Fig. 9) for various positions of these points and show that the expression for d holds in all cases.

Show that the distance between two points $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ when the axes are oblique, with angle ω, is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + 2(x_2 - x_1)(y_2 - y_1)\cos\omega}$$
$$= \sqrt{(\Delta x)^2 + (\Delta y)^2 + 2\,\Delta x \cdot \Delta y \cdot \cos\omega}.$$

10. Ratio of Division. *If two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ are given by their coordinates, the coordinates x, y of any point P on the line P_1P_2 can be found if the division ratio $P_1P/P_1P_2 = k$ is known in which the point P divides the segment P_1P_2.* Let Q_1, Q_2, Q (Fig. 10), be the projections of P_1, P_2, P on the axis Ox; then the point Q divides Q_1Q_2 in the same ratio k in which P divides P_1P_2. Now as $OQ_1 = x_1$, $OQ_2 = x_2$, $OQ = x$, it follows from § 3 that

$$x = x_1 + k(x_2 - x_1).$$

In the same way we find by projecting P_1, P_2, P on the axis Oy that

$$y = y_1 + k(y_2 - y_1).$$

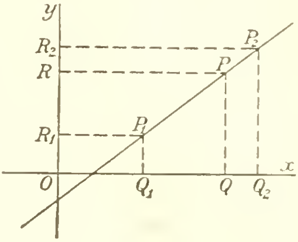


FIG. 10

Thus, the coordinates x, y of P are found expressed in terms of the coordinates of P_1, P_2 and the division ratio k. Putting again $x_2 - x_1 = \Delta x$, $y_2 - y_1 = \Delta y$, we may also write

$$x = x_1 + k \cdot \Delta x, \quad y = y_1 + k \cdot \Delta y.$$

Here again the student should convince himself that the formulas hold generally for any position of the two points, by selecting numerous examples. He should also prove, from a figure, that the same expressions for the coordinates of the point P hold for oblique coordinates.

As in § 3, if the division ratio k is negative, the two segments P_1P_2 and P_1P must have opposite sense, so that the points P and P_2 must lie on opposite sides of the point P_1.

Find, *e.g.*, the coordinates of the points that divide the segment joining $(-4, 3)$ to $(6, -5)$ in the division ratios $k = \frac{1}{3}$, $k = 2$, $k = -\frac{1}{2}$, $k = -1$, and indicate the four points in a figure.

11. Midpoint of a Segment. *The midpoint P of a segment P_1P_2 has for its coordinates the arithmetic means of the corresponding coordinates of P_1 and P_2*: that is, if x_1, y_1 are the coordinates of P_1, x_2, y_2 those of P_2, the division ratio being $k = \frac{1}{2}$, the coordinates of the midpoint P are (§ 10)

$$x = x_1 + \tfrac{1}{2}(x_2 - x_1) = \tfrac{1}{2}(x_1 + x_2),$$
$$y = y_1 + \tfrac{1}{2}(y_2 - y_1) = \tfrac{1}{2}(y_1 + y_2).$$

EXERCISES

1. With reference to the same set of axes, locate the points (6, 4), $(2, -\frac{1}{2})$, (− 6.4, − 3.2), (− 4, 0), (− 1, 5), (.001, − 4.01).

2. Locate the points (− 3, 4), (0, − 1), $(6, -\sqrt{2})$, $(\frac{2}{3}, -10\frac{1}{2})$, $(0, a)$, (a, b), (3, − 2), $(-2, \sqrt{2})$.

3. If a and b are positive numbers, in what quadrants do the following points lie: $(a, -b)$, (b, a), (a, a), $(-b, b)$, $(-b, -a)$?

4. Show that the points (a, b) and $(a, -b)$ are symmetric with respect to the axis Ox; that (a, b) and $(-a, b)$ are symmetric with respect to the axis Oy; that (a, b) and $(-a, -b)$ are symmetric with respect to the origin.

5. In the city of Washington the lettered streets (A street, B street, etc.) run east and west, the numbered streets (1st street, 2d street, etc.) north and south, the Capitol being the origin of coordinates. The axes of coordinates are called avenues; thus, *e.g.*, 1st street north runs one block north of the Capitol. If the length of a block were 1/10 mile, what would be the distance from the corner of South C street and East 5th street to the corner of North Q street and West 14th street?

6. Prove that the points (6, 2), (0, − 6), (7, 1) lie on a circle whose center is (3, − 2).

7. A square of side s has its center at the origin and diagonals coincident with the axes; what are the coordinates of the vertices? of the midpoints of the sides?

8. If a point moves parallel to the axis Oy, which of its coordinates remains constant?

9. In what quadrants can a point lie if its abscissa is negative? its ordinate positive?

10. Find the coordinates of the points which trisect the distance between the points $(1, -2)$ and $(-3, 4)$.

11. To what point must the line segment drawn from $(2, -3)$ to $(-3, 5)$ be extended so that its length is doubled? trebled?

12. The abscissa of a point is -3, its distance from the origin is 5; what is its ordinate?

13. A rectangular house is to be built on a corner lot, the front, 30 ft. wide, cutting off equal segments on the adjoining streets. If the house is 20 ft. deep, find the coordinates (with respect to the adjoining streets) of the back corners of the house.

14. A baseball diamond is 90 ft. square and pitcher's plate is 60 ft. from home plate. Using the foul lines as axes, find the coordinates of the following positions:

(*a*) pitcher's plate;

(*b*) catcher 8 ft. back of home plate and in line with second base;

(*c*) base runner playing 12 ft. from first base;

(*d*) third baseman playing midway between pitcher's plate and third base (before a bunt);

(*e*) right fielder playing 90 ft. from first and second base each.

15. How far does the ball go in Ex. 14 if thrown by third baseman in position (*d*) to second base?

16. If right fielder (Ex. 14) catches a ball in position (*e*) and throws it to third base for a double play, how far does the ball go?

17. A park 600 ft. long and 400 ft. wide has six lights arranged in a circle about a central light cluster. All the lights are 200 ft. apart, and the central cluster and two others are in a line parallel to the length of the park. What are the coordinates of all the lights with respect to two boundary hedges?

18. With respect to adjoining walks, three trees have coordinates (30 ft., 8 ft.), (20 ft., 45 ft.), (– 27 ft., 14 ft.), respectively. A tree is to be planted to form the fourth vertex of a parallelogram; where should it be placed? (Three possible positions; best found by division ratio.)

12. Area of a Triangle with One Vertex at the Origin. Let one vertex of a triangle be the origin, and let the other vertices be P_1 (x_1, y_1) and P_2 (x_2, y_2). Draw through P_1 and P_2 lines parallel to the axes (Fig. 11). The area A of the triangle is then obtained by subtracting from the area of the circumscribed rectangle the areas of the three non-shaded triangles; *i.e.*

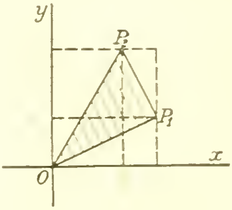


Fig. 11

$$A = x_1y_2 - \tfrac{1}{2}x_1y_1 - \tfrac{1}{2}x_2y_2 - \tfrac{1}{2}(x_1 - x_2)(y_2 - y_1)$$
$$= \tfrac{1}{2}(x_1y_2 - x_2y_1).$$

This formula gives the area with the sign + or − according as the sense of the motion around the perimeter OP_1P_2O is ***counterclockwise*** (opposite to the rotation of the hands of a clock) or ***clockwise***.

For numerical computation it is most convenient to write down the coordinates of the two points thus:

$$\begin{matrix} x_1 & y_1 \\ x_2 & y_2 \end{matrix}$$

and to take half the difference of the crosswise products. The formula is therefore often written in the form

$$A = \frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix},$$

where the symbol $\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$

stands for $x_1y_2 - x_2y_1$, and is called a ***determinant*** (of the second order).

Thus, the area of the triangle formed by the origin with the pair of points (4, 3) and (2, 5) is

$$\frac{1}{2}\begin{vmatrix} 4 & 3 \\ 2 & 5 \end{vmatrix} = \tfrac{1}{2}(4 \times 5 - 2 \times 3) = 7.$$

13. Translation of Axes. Instead of the origin O and the axes Ox, Oy (Fig. 12), let us select a new origin O' (read: O prime) and new axes $O'x'$, $O'y'$, parallel to the old axes. Then any point P whose coordinates with reference to the old axes are $OQ = x$, $QP = y$ will have with reference to the new axes the coordinates $O'Q' = x'$, $Q'P = y'$; and the figure shows that if h, k are the coordinates of the new origin, then

$$x = x' + h,$$
$$y = y' + k.$$

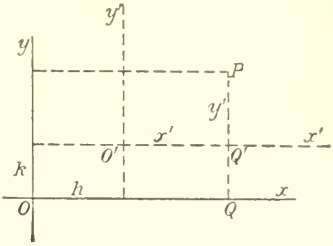


FIG. 12

The change from one set of axes to a new set is called a *transformation of coordinates.* In the present case, where the new axes are parallel to the old, this transformation can be said to consist in a *translation of the axes.*

14. Area of Any Triangle. Let $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, $P_3(x_3, y_3)$ be the vertices of the triangle (Fig. 13). If we take one of these vertices, say P_3, as new origin, with the new axes parallel to the old, the new coordinates of P_1, P_2 will be:

$$x'_1 = x_1 - x_3, \quad x'_2 = x_2 - x_3,$$
$$y'_1 = y_1 - y_3, \quad y'_2 = y_2 - y_3.$$

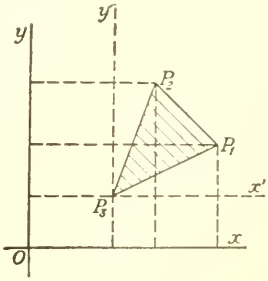


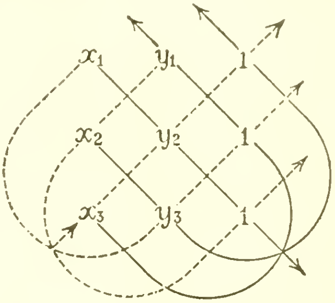
FIG. 13

Hence, by § 12, the area of the triangle $P_1P_2P_3$ is

$$A = \tfrac{1}{2}(x'_1y'_2 - x'_2y'_1) = \tfrac{1}{2}[(x_1 - x_3)(y_2 - y_3) - (x_2 - x_3)(y_1 - y_3)]$$
$$= \tfrac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)].$$

For numerical computation it is best to put down the coordinates of the three points with a 1 after each pair, thus:

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$



Then add the three products formed by following the full lines and subtract the three products formed by following the dotted lines as indicated in the accompanying scheme, *i.e.* form the ***determinant*** (of the third order)

$$= x_1y_2 + x_2y_3 + x_3y_1 - x_3y_2 - x_2y_1 - x_1y_3.$$

This is equal to the expression in the square brackets above, *i.e.* to 2 A. Therefore

$$A = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

Here as in § 12 the sign of the area is + or − according as the sense of the motion along the perimeter $P_1P_2P_3P_1$ is counterclockwise or clockwise.

EXERCISES

1. Find the areas of the triangles having the following vertices:

(*a*) (1, 3), (5, 2), (4, 6); (*b*) (− 2, 1), (2, − 3), (0, − 6);

(*c*) (*a*, *b*), (*a*, 0), (0, *b*); (*d*) (4, 3), (6, − 2), (− 1, 5).

2. Show that the area of the triangle whose vertices are (7, − 8), (− 3, 2), (− 5, − 4) is four times the area of the triangle formed by joining the midpoints of the sides.

3. Find the area of the quadrilateral whose vertices are (2, 3), (− 1, − 1), (− 4, 2), (− 3, 6).

4. Find the area of the triangle whose vertices are (*a*, 0), (0, *b*), (− *c*, − *c*).

5. Find the area of the triangle (1, 4), (3, − 2), (− 3, 16). What does your result show about these points?

6. Find the area of the triangle $(a, b + c)$, $(b, c + a)$, $(c, a + b)$. What does the result show whatever the values of a, b, c?

7. Show that the points $(3, 7)$, $(7, 3)$, $(8, 8)$ are the vertices of an isosceles triangle. What is its area? Show that the same is true for the points (a, b), (b, a), (c, c), whatever a, b, c, and find the area.

8. Find the perimeter of the triangle whose vertices are $(3, 7)$, $(2, -1)$, $(5, 3)$. Is the triangle scalene? What is its area?

15. Statistics. Related Quantities. If pairs of values of two related quantities are given, each of these pairs of values is represented by a point in the plane if the value of one quantity is represented by the abscissa and that of the other by the ordinate of the point. A curved line joining these points gives a vivid idea of the way in which the two quantities change. Statistics and the results of scientific experiments are often represented in this manner.

EXERCISES

1. The population of the United States, as shown by the census reports, is approximately as given in the following table:

YEAR	1790	1800	'10	'20	'30	'40	'50	'60	'70	'80	'90	1900	'10
Millions	4	5	7	10	13	17	23	31	39	50	63	76	92

Mark the points corresponding to the pairs of numbers (1790, 4), (1800, 5), etc., on squared paper, representing the time on the horizontal axis and the population vertically. Connect these points by a curved line.

2. From the figure of Ex. 1, estimate approximately the population of the United States in 1875; in 1905; in 1915.

3. From the figure of Ex. 1, estimate approximately when the population was 25 millions; 60 millions; when it will be 100 millions.

4. Draw a figure to represent the growth of the population of your own State, from the figures given by the Census Reports.

[Other data suitable for statistical graphs can be found in large quantity in the Census Reports; in the Crop Reports of the government; in the quotations of the market prices of food and of stocks and bonds; in the World Almanac; and in many other books.]

5. The temperatures on a certain day varied hour by hour as follows:

	A.M.						N.	P.M.								
Time . .	6	7	8	9	10	11	12	1	2	3	4	5	6	7	8	9
Temp. . .	50	52	55	60	64	67	70	72	74	75	74	72	69	65	60	57

Draw a figure to represent these pairs of values.

6. In experiments on stretching an iron bar, the tension t (in tons) and the elongation E (in thousandths of an inch) were found to be as follows:

t (in tons)	1	2	4	6	8	10
E (in thousandths of an inch) .	10	19	38	60	81	103

Draw a figure to represent these pairs of values.

[Other data can be found in books on Physics and Engineering.]

7. By Hooke's law, the elongation E of a stretched rod is supposed to be connected with the tension t by the formula $E = c \cdot t$, where c is a constant. Show that if $c = 10$, with the units of Ex. 6, the values of E and t would be nearly the same as those of Ex. 6. Plot the values given by the formula and compare with the figure of Ex. 6.

8. The distances through which a body will fall from rest in a vacuum in a time t are given by the formula $s = 16\,t^2$, approximately, if t is in seconds and s is in feet. Show that corresponding values of s and t are

t	1	2	3	4	5	6
s	16	64	144	256	400	576

Draw a figure to represent these pairs of values.

16. Polar Coordinates. The position of a point P in a plane (Fig. 14) can also be assigned by its distance $OP = r$ from a fixed point, or ***pole***, O, and the angle $xOP = \phi$, made by the line OP with a fixed line Ox, the ***polar axis***. The distance r is called the ***radius vector***, the angle ϕ the ***polar angle*** (or also the *vectorial angle*, *azimuth*, *amplitude*, or *anomaly*), of the point P. The radius vector r and the polar angle ϕ are called the ***polar coordinates*** of P.

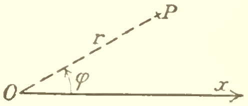


Fig. 14

Locate the points: $(5, \frac{1}{3}\pi)$, $(6, \frac{5}{3}\pi)$, $(2, 140°)$, $(7, 307°)$, $(\sqrt{5}, \pi)$, $(4, 0°)$.

To obtain for every point in the plane a single definite pair of polar coordinates it is *sufficient* to take the radius vector r always positive and to regard as polar angle the positive angle between 0 and 2π $(0 \leqq \phi < 2\pi)$ through which the polar axis (regarded as a half-line or ray issuing from the pole O) must be turned about the pole O in the counterclockwise sense to pass through P. The only exception is the pole O for which $r = 0$, while the polar angle is indeterminate.

But it is not *necessary* to confine the radius vector to positive values and the polar angle to values between 0 and 2π. A single definite point P will correspond to every pair of real values of r and ϕ, if we agree that a negative value of the radius vector means that the distance r is to be laid off in the negative sense on the polar axis, after being turned through the angle ϕ, and that a negative value of ϕ means that the polar axis should be turned in the clockwise sense.

The polar angle is then not changed by adding to it any positive or negative integral multiple of 2π; and a point whose polar coordinates are r, ϕ can also be described as having the coordinates $-r$, $\phi \pm \pi$.

Locate the points:

$$(3, -\tfrac{1}{2}\pi),\ (a, -\tfrac{2}{3}\pi),\ (-5, 75°),\ (-3, -20°).$$

17. Transformation from Cartesian to Polar Coordinates, and *vice versa*. The coordinates $OQ = x$, $QP = y$, defined in § 4, are called ***cartesian*** coordinates, to distinguish them from the polar coordinates. The term is derived from the Latin form, Cartesius, of the name of René Descartes, who first applied the method of coordinates systematically (1637), and thus became the founder of analytic geometry.

The relation between the cartesian and polar coordinates of one and the same point P appears from Fig. 15. We have evidently:

$$\begin{cases} x = r\cos\phi, \\ y = r\sin\phi, \end{cases} \quad \text{and} \quad \begin{cases} r = \sqrt{x^2 + y^2}, \\ \tan\phi = \dfrac{y}{x}. \end{cases}$$

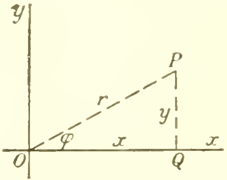


Fig. 15

18. Distance between Two Points in Polar Coordinates.

If two points P_1, P_2 are given by their polar coordinates, r_1, ϕ_1 and r_2, ϕ_2, the distance $d = P_1P_2$ between them is found from the triangle OP_1P_2 (Fig. 16), by the cosine law of trigonometry, if we observe that the angle at O is equal to $\pm(\phi_2 - \phi_1)$:

$$d = \sqrt{r_1^2 + r_2^2 - 2\,r_1r_2\cos(\phi_2 - \phi_1)}.$$

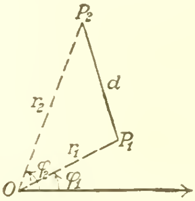


Fig. 16

EXERCISES

1. Find the distances between the points: $(2, \frac{1}{6}\pi)$ and $(4, \frac{3}{4}\pi)$; $(a, \frac{1}{2}\pi)$ and $(3a, \frac{1}{6}\pi)$.

2. Find the cartesian coordinates of the points $(5, \frac{1}{4}\pi)$, $(6, -\frac{1}{4}\pi)$, $(4, \frac{1}{3}\pi)$, $(2, \frac{2}{3}\pi)$, $(7, \pi)$, $(6, -\pi)$, $(4, 0)$, $(-3, 60^\circ)$, $(-5, -90^\circ)$.

3. Find the polar coordinates of the points $(\sqrt{3}, 1)$, $(-\sqrt{3}, 1)$, $(1, -1)$, $(-\frac{1}{2}, -\frac{1}{2})$, $(-a, a)$.

4. Find an expression for the area of the triangle whose vertices are $(0, 0)$, (r_1, ϕ_1), and (r_2, ϕ_2).

5. Find the area of the triangle whose vertices are (r_1, ϕ_1), (r_2, ϕ_2), (r_3, ϕ_3).

6. Find the radius vector of the point P on the line joining the points $P_1(r_1, \phi_1)$ and $P_2(r_2, \phi_2)$ such that the polar angle of P is $\frac{1}{2}(\phi_1 + \phi_2)$.

7. If the axes are oblique with angle ω, what are the relations existing between the cartesian and polar coordinates of a point?

19. Projection of Vectors. A straight line segment AB of definite length, direction, and *sense* (indicated by an arrowhead, pointing from A to B) is called a ***vector.*** The ***projection*** $A'B'$ (Figs. 17, 18) *of a vector AB on an* ***axis,*** *i.e.* on a line l on which a definite sense has been selected as positive, *is the product of the length* (or absolute value) *of the vector AB into the cosine of the angle between the positive senses of the axis and the vector:*

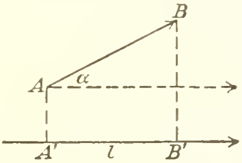


FIG. 17

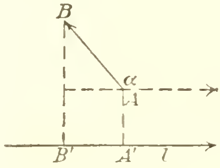


FIG. 18

$$A'B' = AB \cos \alpha.$$

The positive sense of the axis (drawn through the initial point of the vector) makes with the vector two angles whose sum is $2\pi = 360°$. As their cosines are the same it makes no difference which of the two angles is used.

With these conventions it is readily seen that *the sum of the projections of the sides of an open polygon on any axis is equal to the projection of the closing side on the same axis,* the sides of the open polygon being taken in the same sense around the perimeter. Thus, in Fig. 19,

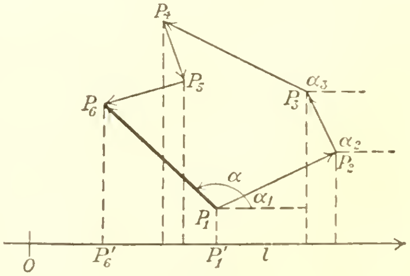


FIG. 19

the vectors P_1P_2, P_2P_3, $\cdots$ P_5P_6 are inclined at the angles α_1, α_2, $\cdots$ α_5 to the axis l; the closing line P_1P_6 makes the angle α with l; its projection is $P'_1P'_6$; and we have

$$P_1P_2 \cos \alpha_1 + P_2P_3 \cos \alpha_2 + P_3P_4 \cos \alpha_3 + P_4P_5 \cos \alpha_4 + P_5P_6 \cos \alpha_5$$
$$= P'_1P'_6 = P_1P_6 \cos \alpha.$$

For, if the abscissas of P_1, P_2, $\cdots$ P_6 measured along l, from any origin O on l, are x_1, x_2, $\cdots$ x_6, the projections of the vectors are $x_2 - x_1$, $x_3 - x_2$, etc., so that our equation becomes the identity:

$$x_2 - x_1 + x_3 - x_2 + x_4 - x_3 + x_5 - x_4 + x_6 - x_5 = x_6 - x_1.$$

20. Components and Resultants of Vectors. In physics, forces, as well as velocities, accelerations, etc., are represented by vectors because such magnitudes have not only a numerical value but also a definite direction and sense.

According to the *parallelogram law* of physics, two forces OP_1, OP_2, acting on the same particle, are together equivalent to the single force OP (Fig. 20), whose vector is the diagonal of the parallelogram formed with OP_1, OP_2 as adjacent sides. The same law holds for simultaneous velocities and accelerations, and for simultaneous or consecutive rectilinear translations. The vector OP is called the ***resultant*** of OP_1 and OP_2, and the vectors OP_1, OP_2 are called the ***components*** of OP.

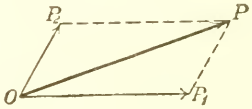


FIG. 20

To construct the resultant it suffices to lay off from the extremity of the vector OP_1 the vector $P_1P = OP_2$; the closing line OP is the resultant. This leads at once to finding the resultant OP of any number of vectors, by *adding* the component vectors *geometrically*, *i.e.* putting them together endwise successively, as in Fig. 21, where the dotted lines need not be drawn.

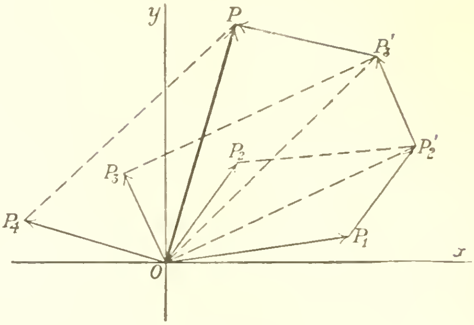


FIG. 21

By § 19, the projection of the resultant on any axis is equal to the sum of the projections of all the components on the same axis.

EXERCISES

1. The cartesian coordinates x, y of any point P are the projections of its radius vector OP on the axes Ox, Oy. (See § 16.)

2. The projection of any vector AB on the axis Ox is the difference of the abscissas of A and B; similarly for Oy.

3. A force of 10 lb. is inclined to the horizon at 60°; find its horizontal and vertical components.

4. A ship sails 40 miles N. 60° E., then 24 miles N. 45° E. How far is the ship then from its starting point? How far east? How far north?

5. A point moves 5 ft. along one side of an equilateral triangle, then 6 ft. parallel to the second, and finally 8 ft. parallel to the third side. What is the distance from the starting point?

6. The sum of the projections of the sides of any *closed* polygon on any axis is zero.

7. If three forces acting on a particle are parallel and proportional to the sides of a triangle, the forces are in equilibrium, *i.e.* their resultant is zero. Similarly for any closed polygon.

8. Find the resultant of the forces OP_1, OP_2, OP_3, OP_4, OP_5, if the coordinates of P_1, P_2, P_3, P_4, P_5, with O as origin, are (3, 1), (1, 2), (− 1, 3), (− 2, − 2), (2, − 2). (Resolve each force into its components along the axes.)

9. If any number of vectors (in the same plane), applied at the origin, are given by the coordinates x, y of their extremities, the length of the resultant is $=\sqrt{(\Sigma x)^2+(\Sigma y)^2}$ (where Σx means the sum of the abscissas, Σy the sum of the ordinates), and its direction makes with Ox an angle α such that $\tan\alpha = \Sigma y / \Sigma x$.

10. Find the horizontal and vertical components of the velocity of a ball when moving 200 ft./sec. at an angle of 30° to the horizon.

11. Six forces of 1, 2, 3, 4, 5, 6 lb., making angles of 60° each with the next, are applied at the same point, in a plane; find their resultant.

12. A particle at one vertex of a square is acted upon by three forces represented by the vectors from the particle to the other three vertices; find the resultant.

21. Geometric Propositions. In using analytic geometry to prove general geometric propositions, it is generally convenient to select as origin a prominent point in the geometric figure, and as axes of coordinates prominent lines of the figure. But sometimes greater symmetry and elegance is gained by taking the coordinate system in a general position. (See, *e.g.*, Exs. 14, 17, 18, below.)

MISCELLANEOUS EXERCISES

1. A regular hexagon of side 1 has its center at the origin and one diagonal coincident with the axis Ox: find the coordinates of the vertices.

2. Show by similar triangles that the points (1, 4), (3, − 2), (− 2, 13) lie on a straight line.

3. If a square, with each side 5 units in length, is placed with one vertex at the origin and a diagonal coincident with the axis Ox, what are the coordinates of the vertices?

4. If a rectangle, with two sides 3 units in length and two sides $3\sqrt{3}$ units in length, is placed with one vertex at the origin and a diagonal along the axis Ox, what are the coordinates of the vertices? There are two possible positions of the rectangle; give the answers in both cases.

5. Show that the points $(0, -1)$, $(-2, 3)$, $(6, 7)$, $(8, 3)$ are the vertices of a parallelogram. Prove that this parallelogram is a *rectangle*.

6. Show that the points $(1, 1)$, $(-1, -1)$, $(+\sqrt{3}, -\sqrt{3})$ are the vertices of an equilateral triangle.

7. Show that the points $(6, 6)$, $(3/2, -3)$, $(-3, 12)$, $(-\frac{15}{2}, 3)$ are the vertices of a parallelogram.

8. Find the radius and the coordinates of the center of the circle passing through the three points $(2, 3)$, $(-2, 7)$, $(0, 0)$.

9. The vertices of a triangle are $(0, 6)$, $(4, -3)$, $(-5, 6)$. Find the lengths of the medians and the coordinates of the *centroid* of the triangle, *i.e.* of the intersection of the medians.

Prove the following propositions:

10. The diagonals of any rectangle are equal.

11. The distance between the midpoints of two sides of any triangle is equal to half the third side.

12. The distance between the midpoints of the non-parallel sides of a trapezoid is equal to half the sum of the parallel sides.

13. In a right triangle, the distance from the vertex of the right angle to the midpoint of the hypotenuse is equal to half the hypotenuse.

14. The line segments joining the midpoints of the adjacent sides of a quadrilateral form a parallelogram.

15. If two medians of a triangle are equal, the triangle is isosceles.

16. In any triangle the sum of the squares of any two sides is equal to twice the square of the median drawn to the midpoint of the third side plus half the square of the third side.

17. The line segments joining the midpoints of the opposite sides of any quadrilateral bisect each other.

18. The sum of the squares of the sides of a quadrilateral is equal to the sum of the squares of the diagonals plus four times the square of the line segment joining the midpoints of the diagonals.

19. The difference of the squares of any two sides of a triangle is equal to the difference of the squares of their projections on the third side.

20. The vertices (x_1, y_1), (x_2, y_2), (x_3, y_3) of a triangle being given, find the *centroid* (intersection of medians).

CHAPTER II

THE STRAIGHT LINE

22. Line Parallel to an Axis. When the coordinates x, y of a point P with reference to given axes Ox, Oy are known, the position of P in the plane of the axes is determined completely and uniquely. Suppose now that only one of the coordinates is given, say, $x = 3$; what can be said about the position of the point P? It evidently lies somewhere on the line AB (Fig. 22) that is parallel to the axis Oy and has the distance 3 from Oy. Every point of the line AB has an abscissa $x = 3$, and every point whose abscissa is 3 lies on the line AB. For this reason we say that the equation

$$x = 3$$

represents the line AB; we also say that $x = 3$ is *the equation of the line AB.*

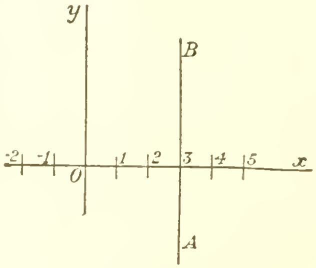


FIG. 22

More generally, the equation $x = a$, where a is any real number, represents that parallel to the axis Oy whose distance from Oy is a. Similarly, the equation $y = b$ represents a parallel to the axis Ox.

EXERCISES

Draw the lines represented by the equations:

1. $x = -2$.

2. $x = 0$.

3. $x = 12.5$.

4. $5x = 7$.

5. $y = 0$.

6. $2y = -7$.

7. $3x + \frac{1}{2} = 0$.

8. $10 - 3y = 0$.

9. $y = \pm 2$.

23. Line through the Origin. Let us next consider any line* through the origin O, such as the line OP in Fig. 23. The points of this line have the property that the ratio y/x of their coordinates is the same, wherever on this line the point P be taken. This ratio is equal to the tangent of the angle α made by the line with the axis Ox, *i.e.* to what we shall call the ***slope*** of the line. Let us put

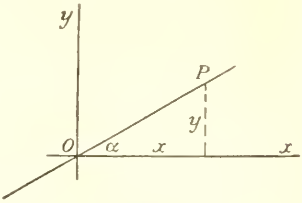


FIG. 23

$$\tan\alpha = m;$$

then we have, for any point P on this line: $y/x = m$, *i.e.*:

$$(1) \qquad y = mx.$$

Moreover, for any point Q not on this line, the ratio y/x must evidently be different from $\tan\alpha$, *i.e.* from m. The equation $y = mx$ is therefore said to *represent the line through O whose slope is m;* and $y = mx$ is called *the equation of this line.* We mean by this statement that the relation $y = mx$ is satisfied by the coordinates of every point on the line OP, and only by the coordinates of the points on this line. Notice in particular that the coordinates of the origin O, *i.e.* $x = 0$, $y = 0$, satisfy the equation $y = mx$.

24. Proportional Quantities. Any two values of x are *proportional* to the corresponding values of y if $y = mx$. For, if (x_1, y_1) and (x_2, y_2) are two pairs of values of x and y that satisfy (1), we have

$$y_1 = mx_1, \quad y_2 = mx_2;$$

---

* For the sake of brevity, a *straight line* will here in general be spoken of simply as a *line;* a line that is not straight will be called a *curve.*

hence, dividing,

$$y_1/y_2 = x_1/x_2.$$

The constant quantity m is called the *factor of proportionality.*

Many instances occur in mathematics and in the applied sciences of two quantities related to each other in this manner. It is often said that one quantity y *varies as* the other quantity x.

Thus Hooke's Law states that the elongation E of a stretched wire or spring varies as the tension t; that is, $E = kt$, where k is a constant.

Again, the circumference c of a circle varies as the radius r; that is,

$$c = 2\pi r.$$

EXERCISES

1. Draw each of the lines:

(*a*) $y = 2x$. (*c*) $y = -\frac{7}{12}x$. (*e*) $5x + 3y = 0$. (*g*) $y = -x$.
(*b*) $y = -3x$. (*d*) $5y = 3x$. (*f*) $y = x$. (*h*) $x - y = 0$.

2. Show that the equation $ax + by = 0$ can be reduced to the form $y = mx$, if $b \neq 0$, and therefore represents a line through the origin.

3. Find the slope of the lines:

(*a*) $x + y = 0$. (*c*) $3x - \frac{1}{10}y = 0$.
(*b*) $x - y = 0$. (*d*) $\sqrt{2}x + y = 0$.

4. Draw a line to represent Hooke's Law $E = kt$, if $k = 10$ (see Ex. 7, p. 15). Let t be represented as horizontal lengths (as is x in § 23) and let E be represented by vertical lengths (as is y in § 23).

5. Draw a line to represent the relation $c = 2\pi r$, where c means the circumference and r the radius of a circle.

6. The number of yards y in a given length varies as the number of feet f in the same length; in particular, $f = 3y$. Draw a figure to represent this relation.

7. If 1 in. = 2.54 cm., show that $c = 2.54\,i$, where c is the number of centimeters and i is the number of inches in the same length. Draw a figure.

25. Slope Form. Finally, consider a line that does not pass through the origin and is not parallel to either of the axes of coordinates (Fig. 24); let it intersect the axes Ox, Oy at A, B, respectively, and let $P(x, y)$ be any other point on it. The figure shows that the *slope m of the line, i.e.* the tangent of the angle α at which the line is inclined to the axis Ox, is

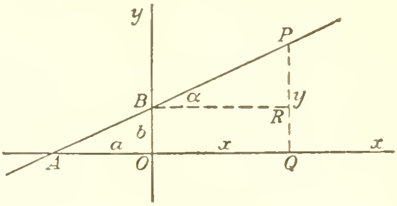


FIG. 24

$$m = \tan \alpha = \frac{RP}{BR};$$

or, since $RP = QP - QR = QP - OB = y - b$ and $BR = OQ = x$:

$$m = \frac{y - b}{x};$$

that is,

$$y = mx + b, \tag{2}$$

where $b = OB$ is called the ***intercept*** made by the line on the axis Oy, or briefly the ***y-intercept.***

The *slope angle* α at which the line is inclined to the axis Ox is always understood as the smallest angle through which the positive half of the axis Ox must be turned counterclockwise about the origin to become parallel to the line.

26. Equation of a Line. On the line AB of Fig. 24 take any other point P'; let its coordinates be x', y', and show that

$$y' = mx' + b.$$

Take the point P' (x', y') outside the line AB and show that the equation $y = mx + b$ is *not* satisfied by the coordinates x', y' of such a point.

For these reasons the equation $y = mx + b$ is said to *represent the line whose y-intercept is b and whose slope is m*; it is also called *the equation of this line.* The y-intercept $OB = b$ and the slope $m = \tan \alpha$ together fully determine the line.

Every line of the plane can be represented by an equation of the form

$$y = mx + b,$$

excepting the lines parallel to the axis Oy. When the line becomes parallel to the axis Oy, both its slope m and its y-intercept b become infinite. We have seen in § 22 that the equation of a line parallel to the axis Oy is of the form $x = a$.

Reduce the equation $3x - 2y = 5$ to the form $y = mx + b$ and sketch the line.

EXERCISES

1. Sketch the lines whose y-intercept is $b = 2$ and whose slopes are $m = \frac{1}{2}$, 3, 0, $-\frac{2}{3}$; write down their equations.

2. Sketch the lines whose slope is $m = 4/3$ and whose y-intercepts are 0, 1, 2, 5, -1, -2, -6, -12.2, and write down their equations.

3. Sketch the lines whose equations are:

(*a*) $y = 2x + 3$. (*c*) $y = x - \frac{1}{2}$. (*e*) $x - y = 1$. (*g*) $7x - y + 12 = 0$.

(*b*) $y = -\frac{1}{2}x + 1$. (*d*) $x + y = 1$. (*f*) $x - 2y + 2 = 0$. (*h*) $4x + 3y + 5 = 0$.

4. Do the points (1, 5), (−2, −1), (3, 7) lie on the line $y = 2x + 3$?

5. A cistern that already contained 300 gallons of water is filled at the rate of 100 gallons per hour. Show that the amount A of water in the cistern n hours after filling begins is $A = 100n + 300$. Draw a figure to represent this relation, plotting the values of A vertically, with 1 vertical space = 100 gallons.

6. In experiments with a pulley block, the pull p in lbs., required to lift a load l in lbs., was found to be expressed by the equation $p = .15l + 2$. Draw this line. How much pull is required to operate the pulley with no load (*i.e.* when $l = 0$)?

7. The readings of a gas meter being tested, T, were found in comparison with those of a standard gas meter S, and the two readings satisfied the equation $T = 300 + 1.2S$. Draw a figure. What was the reading T when the reading S was zero? What is the meaning of the slope of the line in the figure?

27. Parallel and Perpendicular Lines. Two lines

$$y = m_1x + b_1, \quad y = m_2x + b_2$$

are obviously ***parallel*** if they have the same slope, *i.e.* if

$$m_1 = m_2. \tag{3}$$

Two lines $y = m_1x + b_1$, $y = m_2x + b_2$ are ***perpendicular*** if the slope of one is equal to minus the reciprocal of the slope of the other, *i.e.* if

$$m_1m_2 = -1. \tag{4}$$

For if $m_1 = \tan\alpha_1$, $m_2 = \tan\alpha_2$, the condition that $m_1m_2 = -1$ gives $\tan\alpha_2 = -1/\tan\alpha_1 = -\cot\alpha_1$, whence $\alpha_2 = \alpha_1 + \frac{1}{2}\pi$.

EXERCISES

1. Write down the equation of any line: (*a*) parallel to $y = 3x - 2$, (*b*) perpendicular to $y = 3x - 2$.

2. Show that the parallel to $y = 3x - 2$ through the origin is $y = 3x$.

3. Show that the perpendicular to $y = 3x - 2$ through the origin is

$$y = -\tfrac{1}{3}x.$$

4. For what value of b does the line $y = 3x + b$ pass through the point (4, 1)? Find the parallel to $y = 3x - 2$ through the point (4, 1).

5. Find the parallel to $y = 5x + 1$ through the point (2, 3).

6. Find the perpendicular to $y = 2x - 1$ through the point (1, 4).

7. What is the geometrical meaning of $b_1 = b_2$ in the equations

$$y = m_1x + b_1, \quad y = m_2x + b_2?$$

8. Two water meters are attached to the same water pipe and the water is allowed to flow steadily through the pipe. The readings R_1 and R_2 of the two meters are found to be connected with the time t by means of the equations

$$R_1 = 2.5t, \quad R_2 = 2.5t + 150,$$

where R_1 and R_2 are measured in cubic feet and t is measured in seconds. Show that the lines that represent these equations are parallel. What is the meaning of this fact?

9. The equations connecting the pull p required to lift a load w is found for two pulley blocks to be

$$p_1 = .05w + 2, \quad p_2 = .05w + 1.5$$

Show that the lines representing these equations are parallel. Explain.

10. The equations connecting the pull p required to lift a load w is found for two pulley blocks to be

$$p_1 = .15\,w + 1.5, \quad p_2 = .05\,w + 1.5.$$

Show that the lines representing these equations are not parallel, but that the values of p_1 and p_2 are equal when $w = 0$. Explain.

28. Linear Function. The equation $y = mx + b$, when m and b are given, assigns to every value of x one and only one definite value of y. This is often expressed by saying that $mx + b$ is a ***function*** of x; and as the expression $mx + b$ is of the first degree in x, it is called a *function of the first degree* or, owing to its geometrical meaning, a ***linear function*** of x.

Examples of functions of x that are *not linear* are $3x^2 - 5$, $ax^2 + bx + c$, $x(x-1)$, $1/x$, $\sin x$, 10^x, etc. The equations $y = 3x^2 - 5$, $y = ax^2 + bx + c$, etc., represent, as we shall see later, not straight lines but curves.

The linear function $y = mx + b$, being the most simple kind of function, occurs very often in the applications. Notice that the constant b is the value of the function for $x = 0$. The constant m is the ***rate of change*** of y with respect to x.

29. Illustrations. Example 1. A man, on a certain date, has $10 in bank; he deposits $3 at the end of every week; how much has he in bank x weeks after date?

Denoting by y the number of dollars in bank, we have

$$y = 3x + 10.$$

His deposit at any time x is a linear function of x. Notice that the coefficient of x gives the ***rate of increase*** of this deposit; in the graph this is the ***slope*** of the line.

Example 2. Water freezes at 0° C. and 32° F.; it boils at 100° C. and at 212° F.; assuming that mercury expands uniformly, *i.e.* proportionally to the temperature, and denoting by x any temperature in Centigrade degrees, by y the same temperature in Fahrenheit degrees, we have

$$\frac{y-32}{x}=\frac{212-32}{100}=\frac{9}{5},\ \text{i.e. } y=\tfrac{9}{5}x+32.$$

If the line represented by this equation be drawn accurately, on a sufficiently large scale, it could be used to convert centigrade temperature into Fahrenheit temperature, and *vice versa*.

EXAMPLE 3. A rubber band, 1 ft. long, is found to stretch 1 in. by a suspended mass of 1 lb. Let the suspended mass be increased by 1 oz., 2 oz., etc., and let the corresponding lengths of the band be measured. Plotting the masses as abscissas and the lengths of the band as ordinates, it will be found that the points (x, y) lie very nearly on a straight line whose equation is $y=\frac{1}{12}x+1$. The experimental fact that the points lie on a straight line, *i.e.* that the function is linear, means that the *extension*, $y-1$, is proportional to the *tension*, *i.e.* to the weight of the suspended mass x (Hooke's Law).

Notice that only the part of the line in the first quadrant, and indeed only a portion of this, has a physical meaning. Can this range be extended by using a spiral steel spring?

EXAMPLE 4. When a point P moves along a line so as to describe always equal spaces in equal times, its motion is called *uniform*. The spaces passed over are then proportional to the times in which they are described, and the coefficient of proportionality, *i.e.* the ratio of the distance to the time, is called the *velocity* v of the uniform motion. If at the time $t=0$ the moving point is at the distance s_0, and at the time t at the distance s, from the origin, then

$$s=s_0+vt.$$

Thus, in uniform motion, the distance s is a linear function of the time t, and the coefficient of t is the speed: $v=(s-s_0)/t$.

EXAMPLE 5. When a body falls from rest (in a vacuum) its velocity v is proportional to the time t of falling: $v = gt$, where g is about 32 if the velocity is expressed in ft./sec., or 980 if the velocity is expressed in cm./sec.

If, at the time $t = 0$, the body is thrown downward with an initial velocity v_0, its velocity at any subsequent time t is

$$v = v_0 + gt.$$

Thus the velocity is a linear function of t, and the coefficient g of t denotes the *rate* at which the velocity changes with the time, *i.e.* the *acceleration* of the falling body.

EXERCISES

1. Draw the line represented by the equation $y = \frac{9}{5}x + 32$ of Example 2, § 29. What is its slope? What is the y-intercept? What is the meaning of each of these quantities if y and x represent the temperatures in Fahrenheit and in Centigrade measure, respectively?

2. Represent the equation $y = \frac{1}{12}x + 1$ of Example 3, § 29, by a figure. What is the meaning of the y-intercept?

3. Draw the line $s = s_0 + vt$ of Example 4, § 29, for the values $s_0 = 10$, $v = 3$. What is the meaning of v? Show that the speed v may be thought of as the rate of increase of s per second.

4. If, in the preceding exercise, v be given a value greater than 3, how does the new line compare with the one just drawn?

5. If, in Ex. 3, v is given the value 3, and s_0 several different values, show that the lines represented by the equation are parallel. Explain.

6. In experiments on the temperatures at various depths in a mine, the temperature (Centigrade) T was found to be connected with the depth d by the equation $T = 60 + .01\,d$, where d is measured in feet. Draw a figure to represent this equation. Show that the rate of increase of the temperature was 1° per hundred feet.

7. In experiments on a pulley block, the pull p (in lb.) required to lift a weight w (in lb.) was found to be $p = .03\,w + 0.5$. Show that the rate of increase of p is 3 lb. per hundred weight increase in w.

30. General Linear Equation. The equation

$$Ax + By + C = 0,$$

in which A, B, C are any real numbers, is called *the general equation of the first degree* in x and y. The coefficients A, B, C are called the constants of the equation; x, y are called the variables. It is assumed that A and B are not both zero. The terms Ax and By are of the first degree; the term C is said to be of degree zero because it might be written in the form Cx^0; this term C is also called the *constant term.*

Every equation of the first degree,

$$(5) \qquad \boldsymbol{Ax + By + C = 0},$$

in which A and B are not both zero, represents a straight line; and conversely, every straight line can be represented by such an equation. For this reason, every equation of the first degree is called a ***linear equation.***

The first part of this fundamental proposition follows from the fact that, when B is not equal to zero, the equation can be reduced to the form $y = mx + b$ by dividing both sides by B; and we know that $y = mx + b$ represents a line (§ 25). When B is equal to zero, the equation reduces to the form $x = a$, which also represents a line (§ 22).

The second part of the theorem follows from the fact that the equations which we have found in the preceding articles for any line are all particular cases of the equation

$$Ax + By + C = 0.$$

This equation still expresses the same relation between x and y when multiplied by any constant factor, not zero. Thus, any one of the constants A, B, C, if not zero, can be reduced to 1 by dividing both sides of the equation by this constant. The equation is therefore said to contain only *two* (not three) *essential constants.*

31. Conditions for Parallelism and for Perpendicularity. It is easy to recognize whether two lines whose equations are $Ax + By + C = 0$ and $A'x + B'y + C'' = 0$ are parallel or perpendicular. The lines are parallel if they have the same slope, and they are perpendicular (§ 27) if the product of their slopes is equal to -1. The slopes of our lines are $-A/B$ and $-A'/B'$; hence these lines are *parallel* if $-A/B = -A'/B'$, *i.e.* if

$$A : B = A' : B';$$

and they are *perpendicular* if

$$\frac{A}{B}\cdot\frac{A'}{B'} = -1, \qquad \textit{i.e.}\text{ if} \qquad AA' + BB' = 0.$$

32. Intercept Form. If the constant term C in a linear equation is zero, the equation represents a line through the origin. For, the coordinates (0, 0) of the origin satisfy the equation

$$Ax + By = 0.$$

If the constant term C is not equal to zero, the equation $Ax + By + C = 0$ can be divided by C; it then reduces to the form

$$\frac{A}{C}x + \frac{B}{C}y + 1 = 0.$$

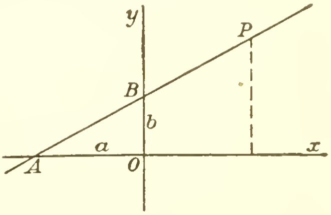


FIG. 25

If A and B are both different from zero, this can be written:

$$\frac{x}{-C/A} + \frac{y}{-C/B} = 1,$$

or putting $-C/A = a$, $-C/B = b$:

$$\frac{x}{a} + \frac{y}{b} = 1. \tag{6}$$

The conditions $A \neq 0$, $B \neq 0$ mean evidently that the line is not parallel to either of the axes. Therefore the equation of any line, not passing through the origin, and not parallel to either axis, can be written in the form (6). With $y = 0$ this equation gives $x = a$; with $x = 0$ it gives $y = b$. Thus

$$a = -\frac{C}{A},\ b = -\frac{C}{B}$$

are the ***intercepts*** (Fig. 25) made by the line on the axes Ox, Oy, respectively (see § 25).

EXERCISES

1. Write down the equations of the line whose intercepts on the axes Ox, Oy are 5 and -3, respectively; the line whose intercepts are $-\frac{3}{2}$ and 7; the line whose intercepts are -1 and $-\frac{7}{2}$. Sketch each of the lines and reduce each of the equations to the form $Ax + By + C = 0$, so that A, B, C are integers.

2. Find the intercepts of the lines: $3x - 2y = 1$, $x + 7y + 1 = 0$, $-3x + \frac{1}{2}y - 5 = 0$. Try to read off the values of the intercepts directly from these equations as they stand.

3. In Ex. 2, find the slopes of the lines.

4. Prove (6), § 32 by equality of areas, after clearing of fractions.

5. What is the equation of the axis Oy? of the axis Ox?

6. What is the value of B such that the line represented by the equation $4x + By - 14 = 0$ passes through the point $(-5, 17)$?

7. What is the value of A such that the line $Ax + 7y = 10$ has its x-intercept equal to -8?

8. Reduce each of the following equations to the intercept form (6), and draw the lines:

(*a*) $3x - 5y - 16 = 0$.
(*b*) $x + \frac{1}{2}y + 7 = 0$.
(*c*) $\dfrac{4x - 3y - 6}{x + y} = 2$.
(*d*) $5x = 3x + y - 10$.

9. Reduce the equations of Ex. 8 to the slope form (2), § 25.

10. Find the equation of the line of slope 6 passing through the point $(6, -5)$.

11. What relation exists between the coefficients of the equation $Ax + By + C = 0$, if the line is parallel to the line $4x - 5y = 8$? parallel to the axis Oy?

12. Show that the points $(-1, -7)$, $(\frac{1}{3}, -3)$, $(2, 2)$, $(-2, -10)$ lie on the same line.

13. Find the area of the triangle formed by the lines $x+y=0$, $x-y=0$, $x-a=0$.

14. Show that the line $4(x-a)+5(y-b)=0$ is perpendicular to the line $5x-4y-10=0$ and passes through the point (a, b).

15. A line has equal positive intercepts and passes through $(-5, 14)$. What is its equation? its slope?

16. If a line through the point $(6, 7)$ has the slope 4, what is its y-intercept? its x-intercept?

17. The Réaumur thermometer is graduated so that water freezes at 0° and boils at 80°. Draw the line that represents the reading R of the Réaumur thermometer as a function of the corresponding reading C of the Centigrade thermometer.

18. What function of the altitude is the area of a triangle of given base?

19. A printer asks 75 ¢ to set the type for a program and 2 ¢ per copy for printing. The total cost is what function of the number of copies printed? Draw the line representing the function.

Another printer asks 3 ¢ per copy, with no charges for setting the type. For how many copies would both charge the same?

20. The sum of two complementary angles α and β is $\frac{1}{2}\pi$; draw the line representing β as a function of α. When $\alpha = \frac{3}{2}\pi$, what is β?

21. Express the value of a note of $ 1000 at the end of the first year as a function of the rate of interest. At 6% simple interest its value is what function of the time in years?

22. Two weights are attached to the opposite ends of a rope that runs through a double pulley block of which one block is fastened at a height above ground. If x and y denote the distances of the two weights above the ground, determine a linear relation between them if $x = 40$ when $y = 0$ and $y = 10$ when $x = 0$.

33. Line through One Point. To find the line of given slope m_1 through a given point $P_1(x_1, y_1)$, observe that the equation must be of the form (2), viz.

$$y = m_1 x + b,$$

since this line has the slope m_1. If this line is to pass through the given point, the coordinates x_1, y_1 must satisfy this equation, *i.e.* we must have

$$y_1 = m_1 x_1 + b.$$

This equation determines b, and the value of b so found might be substituted in the preceding equation. But we can eliminate b more readily between the two equations by subtracting the latter from the former. This gives

$$y - y_1 = m_1(x - x_1)$$

as *the equation of the line of slope* m_1 *through* $P_1(x_1, y_1)$.

The problem of finding *a line through a given point parallel, or perpendicular, to a given line* is merely a particular case of the problem just solved, since the slope of the required line can be found from the equation of the given line (§ 27). If the slope of the given line is $m_1 = \tan \alpha_1$, the slope of any parallel line is also m_1, and the slope of any line perpendicular to it is

$$m_2 = \tan(\alpha_1 + \tfrac{1}{2}\pi) = -\cot \alpha_1 = -\frac{1}{m_1}.$$

34. Line through Two Points. To find the line through two given points, $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, observe (Fig. 26) that the slope of the required line is evidently

$$m_1 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x},$$

if, as in § 9, we denote by Δx, Δy the projections of P_1P_2 on Ox, Oy;

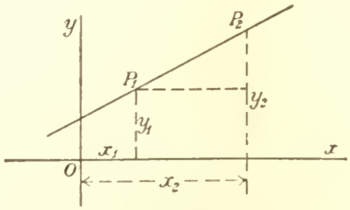


FIG. 26

and as the line is to pass through (x_1, y_1), we find its equation by § 33 as

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1),$$

or

$$y - y_1 = \frac{\Delta y}{\Delta x}(x - x_1).$$

The equation of the line through two given points (x_1, y_1), (x_2, y_2) can also be written in the *determinant form*

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0,$$

which (§ 14) means that the point (x, y) is such as to form with the given points a triangle of zero area. By expanding the determinant it can be shown that this equation agrees with the preceding equation. A more direct proof will be given later (§ 49).

EXERCISES

1. Find the equation of the line through the point $(-7, 2)$ parallel to the line $y = 3x$.

2. Show that the points $(4, -3)$, $(-5, 2)$, $(5, 20)$ are the vertices of a right triangle.

3. Find the equation of the line through the point $(-6, -3)$ which makes an angle of 30° with the axis Ox; 30° with the axis Oy.

4. Does the line of slope $\frac{3}{4}$ through the point $(4, 3)$ pass through the point $(-5, -4)$?

5. Find the equation of the line through the point $(-2, 1)$ parallel to the line through the points $(4, 2)$ and $(-3, -2)$.

6. Find the equations of the lines through the origin which trisect that portion of the line $5x - 6y = 60$ which lies in the fourth quadrant.

7. What are the intercepts of the line through the points $(2, -3)$, $(-5, 4)$?

8. Show that the equation of the line through the point (a, b) perpendicular to the line $Ax + By + C = 0$ is $(x - a)/A = (y - b)/B$.

9. Find the equations of the diagonals of the rectangle formed by the lines $x + a = 0$, $x - b = 0$, $y + c = 0$, $y - d = 0$.

10. Find the equation of the perpendicular bisector of the line joining the points $(4, -5)$ and $(-3, 2)$. Show that any point on it is equally distant from each of the two given points.

11. Find the equation of the line perpendicular to the line $4x - 3y + 6 = 0$ that passes through the midpoint of $(-4, 7)$ and $(2, 2)$.

12. What are the coordinates of a point equidistant from the points $(2, -3)$ and $(-5, 0)$ and such that the line joining the point to the origin has a slope 1?

13. If the axes are oblique with angle ω, show that the slope of the line joining the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is

$$\frac{(y_2 - y_1)\sin\omega}{(x_2 - x_1) + (y_2 - y_1)\cos\omega}.$$

14. If the axes are oblique with angle ω, show that the equation of the line through the point $P_1(x_1, y_1)$ which makes with the axis Ox the angle ϕ, is

$$y - y_1 = \frac{\sin\phi}{\sin(\omega - \phi)}(x - x_1).$$

Is the coefficient of $(x - x_1)$ the slope of this line?

15. In an experiment with a pulley-block it is assumed that the relation between the load l and the pull p required to lift it is linear. Find the relation if $p = 8$ when $l = 100$, and $p = 12$ when $l = 200$.

16. In an experiment in stretching a brass wire it is assumed that the elongation E is connected with the tension t by means of a linear relation. Find this relation if $t = 18$ lb. when $E = .1$ in., and $t = 58$ lb. when $E = .3$ in.

17. A cistern is being filled by water flowing into it at the rate of 30 gallons per second. Assuming that the amount A of water in the cistern is connected with the time t by a linear relation, find this relation if $A = 1000$ when $t = 10$. Hence find A when $t = 0$.

CHAPTER III

SIMULTANEOUS LINEAR EQUATIONS DETERMINANTS

PART I. EQUATIONS IN TWO UNKNOWNS DETERMINANTS OF SECOND ORDER

35. Intersection of Two Lines. *The point of intersection of any two lines is found by solving the equations of the lines as simultaneous equations.* For the coordinates of the point of intersection must satisfy each of the two equations, since this point lies on each of the two lines; and it is the only point having this property. Find the points of intersection of the following pairs of lines:

$$(a)\quad \begin{cases} 4x-3y+3=0, \\ 3x+5y-34=0. \end{cases} \qquad (b)\quad \begin{cases} 3x-5y=0, \\ 7x+2y=0. \end{cases}$$

$$(c)\quad \begin{cases} 2x+y-13=0, \\ 5x-2y+11=0. \end{cases}$$

The solution of simultaneous linear equations is much facilitated by the use of determinants. As, moreover, determinants are used to advantage in many other problems (see, *e.g.*, §§ 12, 14) it is desirable to study determinants systematically before proceeding with the study of the straight line.

36. Solution of Two Linear Equations. To solve two linear equations (§ 30),

$$(1)\qquad \begin{cases} a_1x+b_1y=k_1, \\ a_2x+b_2y=k_2, \end{cases}$$

we may eliminate y to find x, and eliminate x to find y. The elimination of y is done systematically by multiplying the first equation by b_2, the second by b_1, and then subtracting the second from the first; this gives

$$(a_1b_2 - a_2b_1)x = k_1b_2 - k_2b_1.$$

Likewise, to eliminate x, multiply the first equation by a_2, the second by a_1, and subtract the first from the second:

$$(a_1b_2 - a_2b_1)y = a_1k_2 - a_2k_1.$$

If $a_1b_2 - a_2b_1 \neq 0$, we can divide by this quantity and thus find

$$x = \frac{k_1b_2 - k_2b_1}{a_1b_2 - a_2b_1}, \quad y = \frac{a_1k_2 - a_2k_1}{a_1b_2 - a_2b_1}. \tag{2}$$

Observe that the values of x and y are quotients with the same denominator, and that the numerator of x is obtained from this denominator by simply replacing a by k, while the numerator of y is obtained from the same denominator by replacing b by k.

This peculiar form of the numerators and denominators of x and y is brought out more clearly if we agree to write the common denominator $a_1b_2 - a_2b_1$ in the form of a determinant:

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}, \tag{3}$$

as in § 12. Thus

$$\begin{vmatrix} 2 & 3 \\ 7 & 5 \end{vmatrix} = 2 \times 5 - 7 \times 3 = -11;$$

$$\begin{vmatrix} -1 & 7 \\ 4 & 2 \end{vmatrix} = -1 \times 2 - 4 \times 7 = -30.$$

With this notation, the values (2) of x and y are

$$x = \frac{\begin{vmatrix} k_1 & b_1 \\ k_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a_1 & k_1 \\ a_2 & k_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}. \tag{4}$$

37. General Rule. If a, b, c, d are any four numbers, the expression

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix},$$

which stands for $ad - bc$, is called a ***determinant***, more precisely, a determinant *of the second order* because *two* numbers occur in each (horizontal) ***row***, as well as in each (vertical) ***column***. (See § 12.)

The determinant (3) is called the ***determinant of the equations*** (1), § 36.

We can then state the following rule for solving the two linear equations (1): *If the determinant of the equations is not equal to zero, x as well as y is the quotient of two determinants; the denominator is the same,* viz. *the determinant of the equations (1); the numerator of x is obtained from this denominator by replacing the coefficients of x by the constant terms, the numerator of y is found from the same denominator by replacing the coefficients of y by the constant terms.*[*]

EXERCISES

1. Find the values of the following determinants:

(a) $\begin{vmatrix} 10 & 2 \\ 3 & 7 \end{vmatrix}$. (b) $\begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix}$. (c) $\begin{vmatrix} 0 & 1 \\ 5 & 10 \end{vmatrix}$.

(d) $\begin{vmatrix} 0 & 0 \\ 12 & 5 \end{vmatrix}$. (e) $\begin{vmatrix} \frac{1}{2} & -12 \\ \frac{3}{4} & -\frac{2}{5} \end{vmatrix}$. (f) $\begin{vmatrix} a & y \\ -b & x \end{vmatrix}$.

2. Solve the following equations; in writing down the solution, *begin with the denominators:*

(a) $\begin{cases} 3x - 2y = 1, \\ 2x + 3y = 15. \end{cases}$ (b) $\begin{cases} 2x + 7y = 3, \\ 5x - y = -11. \end{cases}$

(c) $\begin{cases} 2x + 3y + 4 = 0, \\ 3x - 5y - 15 = 0. \end{cases}$ (d) $\begin{cases} 5x - 3y - 2 = 0, \\ y = 4x - 1. \end{cases}$

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[*] One great advantage of this rule is that the same rule applies to the solution of any (finite) number of linear equations with the same number of variables. (See § 74.)

38. Exceptions. The process of § 37 cannot be applied when the determinant of the equations (1) vanishes, *i.e.* when

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0,$$

that is, when $a_1 b_2 = a_2 b_1$.

For the sake of simplicity we here assume that none of the four numbers a_1, b_1, a_2, b_2 is zero. If any one of them were zero, we might solve the equation in which it occurs to obtain the value of one of the variables. With this assumption, the condition may be written in the form

$$\frac{a_2}{a_1} = \frac{b_2}{b_1},$$

or, denoting the common value of these quotients by m:

$$a_2 = ma_1, \quad b_2 = mb_1,$$

so that the equations (1) become

$$a_1 x + b_1 y = k_1,$$
$$ma_1 x + mb_1 y = k_2.$$

We must now distinguish two cases, according as $k_2 = mk_1$ or $k_2 \neq mk_1$. In the former case, *i.e.* if

$$k_2 = mk_1,$$

the second equation reduces, upon division by m, to the first equation. Thus, the two equations represent one and the same relation between x and y, and are therefore not sufficient to determine x and y separately. We can assign to either variable an arbitrary value and then find a corresponding value of the other variable. The equations (1) can then be said to have *an infinite number of solutions.*

In the other case, *i.e.* if

$$k_2 \neq mk_1,$$

the equations are evidently inconsistent, and there exist no finite values of x and y satisfying both equations. Thus the equations $\frac{1}{3}x - 2y = 2$, $2x - 12y = 15$ are inconsistent.

39. Geometric Interpretation. All these results about linear equations can be interpreted geometrically. We have seen (§ 30) that every linear equation represents a straight line, and (§ 35) that by solving two such equations we find the coordinates of the point of intersection of the two lines. Now two lines in a plane may either intersect, or coincide, or be parallel. In the first case, they have a single point in common; in the second, they have an infinite number of points in common; in the third, they have no point in common. The first case is that of §§ 36, 37; the last two cases are discussed in § 38. Including the case of coincident lines with that of parallels, we may say that the relation

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0$$

is the necessary and sufficient *condition of parallelism* of the two lines $a_1x + b_1y = k_1, \quad a_2x + b_2y = k_2.$

40. Elimination. If in the linear equations (1) of § 36 the constant terms k_1, k_2 are both zero so that they are

$$a_1x + b_1y = 0,$$
$$a_2x + b_2y = 0,$$

the equations are called *homogeneous*. Obviously, two homogeneous linear equations are always satisfied by the values

$$x = 0, \quad y = 0.$$

If the determinant of the equations does not vanish, *i.e.* if

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0,$$

this solution is also found from § 36, and it is the only solution.

But if

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0,$$

it is found as in § 38 that the equations have an infinite number of solutions. Conversely, *if two homogeneous linear equa-*

tions are satisfied by values of x and y that are not both zero, the determinant of the equations must vanish. For, multiplying the first equation by b_2, and the second by b_1, and subtracting, we find

$$(a_1b_2 - a_2b_1)x = 0.$$

Eliminating x in a similar manner, we find

$$(a_1b_2 - a_2b_1)y = 0.$$

These equations show that unless x and y are both zero we must have

$$a_1b_2 - a_2b_1 = 0, \quad i.e. \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0.$$

This relation is also the result of eliminating x and y between the two equations. For, if, *e.g.*, $x \neq 0$ we may divide both equations by x and then eliminate y/x between the equations

$$a_1 + b_1\frac{y}{x} = 0, \quad a_2 + b_2\frac{y}{x} = 0,$$

by multiplying the former by b_2, the latter by b_1, and subtracting. The result is again $a_1b_2 - a_2b_1 = 0$. Thus *the result of eliminating the variables between two homogeneous linear equations is the determinant of the equations equated to zero.* We shall see later (§ 75) that all the results of the present article are true for any number of homogeneous linear equations.

Geometrically, two homogeneous linear equations of course represent two lines through the origin. The vanishing of the determinant means that the lines coincide so that they have an infinite number of points in common.

EXERCISES

1. Evaluate the determinants:

(*a*) $\begin{vmatrix} 2 & 5 \\ 3 & 4 \end{vmatrix}$; (*b*) $\begin{vmatrix} 7 & -3 \\ 4 & 1 \end{vmatrix}$; (*c*) $\begin{vmatrix} 1 & a \\ -a & 1 \end{vmatrix}$;

(*d*) $\begin{vmatrix} \sin\beta & -\cos\beta \\ \cos\beta & \sin\beta \end{vmatrix}$; (*e*) $\begin{vmatrix} 1 & \cos\beta \\ \cos\beta & 1 \end{vmatrix}$; (*f*) $\begin{vmatrix} a_1 + a_2 & a_2 \\ a_2 & a_2 + a_3 \end{vmatrix}$.

2. Express $x^2 + y^2$ in the form of a determinant of the second order.

3. Verify that

$$\begin{vmatrix} a^2 + b^2 & aa' + bb' \\ aa' + bb' & a'^2 + b'^2 \end{vmatrix} = \begin{vmatrix} a & b \\ a' & b' \end{vmatrix}^2;$$

and that

$$\begin{vmatrix} a^2 + b^2 + c^2 & aa' + bb' + cc' \\ aa' + bb' + cc' & a'^2 + b'^2 + c'^2 \end{vmatrix} = \begin{vmatrix} b & c \\ b' & c' \end{vmatrix}^2 + \begin{vmatrix} c & a \\ c' & a' \end{vmatrix}^2 + \begin{vmatrix} a & b \\ a' & b' \end{vmatrix}^2.$$

4. Verify that

$$\begin{vmatrix} aa' + bb' & ac' + bd' \\ ca' + db' & cc' + dd' \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \cdot \begin{vmatrix} a' & b' \\ c' & d' \end{vmatrix}.$$

5. Find the coordinates of the points of intersection of the following lines; and check by a sketch:

$$(a)\ \begin{cases} 5x - 7y + 11 = 0, \\ 3x + 2y - 12 = 0. \end{cases} \quad (b)\ \begin{cases} 4x + 2y - 7 = 0, \\ 3x - 8y + 4 = 0. \end{cases} \quad (c)\ \begin{cases} 2x - 5y = 3, \\ x + 3y = -1. \end{cases}$$

$$(d)\ \begin{cases} 4x + 2y = 9, \\ 2x - 5y = 0. \end{cases} \quad (e)\ \begin{cases} 3x + 2y = 0, \\ 6x - 4y + 4 = 0. \end{cases} \quad (f)\ \begin{cases} 2.4x + 3.1y = 4.5, \\ .8x + 2y = 6.2. \end{cases}$$

6. Do the following pairs of lines intersect, or are they parallel or coincident?

$$(a)\ \begin{cases} 3x - 6y - 8 = 0, \\ x - 2y + 1 = 0. \end{cases} \quad (b)\ \begin{cases} 3x + y - 6 = 0, \\ x + \tfrac{1}{3}y - 2 = 0. \end{cases} \quad (c)\ \begin{cases} 3x - 5y = 0, \\ 10y - 6x = 0. \end{cases}$$

$$(d)\ \begin{cases} 4x - 2y - 7 = 0, \\ 2x - 3y + 5 = 0. \end{cases} \quad (e)\ \begin{cases} 2x - 6y - 4 = 0, \\ x - 3y - 2 = 0. \end{cases} \quad (f)\ \begin{cases} x + \tfrac{1}{2}y = 0, \\ 2x + 3y = 0. \end{cases}$$

7. For what values of s do the following pairs of lines become parallel?

$$(a)\ \begin{cases} 4x + sy - 15 = 0, \\ 2x - 7y + 10 = 0. \end{cases} \quad (b)\ \begin{cases} 3sx - 8y - 13 = 0, \\ 2x - 2sy + 15 = 0. \end{cases} \quad (c)\ \begin{cases} 7x - 14y + 8 = 0, \\ sx - 2y + s = 0. \end{cases}$$

8. For what values of s do the following pairs of lines coincide?

$$(a)\ \begin{cases} 5x - 7y + 6 = 0, \\ 5x - 7y + s = 0. \end{cases} \quad (b)\ \begin{cases} 3x + 2y + 3 = 0, \\ sx - 2y + s = 0. \end{cases} \quad (c)\ \begin{cases} 3x + 6y - 5 = 0, \\ x + sy - \tfrac{5}{3} = 0. \end{cases}$$

9. Solve the following equations by determinants:

$$(a)\ \begin{cases} U + V = 25, \\ 2U - 3V = 5. \end{cases} \quad (b)\ \begin{cases} x^2 + y^2 = 25, \\ 2x^2 - 3y^2 = 5. \end{cases} \quad (c)\ \begin{cases} s = 16t^2 + 100, \\ 5s + t^2 = 824. \end{cases}$$

$$(d)\ \begin{cases} \dfrac{3}{x} - \dfrac{2}{y} = 2, \\ \dfrac{1}{x} + \dfrac{4}{y} = -\dfrac{5}{3}. \end{cases} \quad (e)\ \begin{cases} \dfrac{1}{x^2} - \dfrac{3}{y^2} = \dfrac{26}{3}, \\ \dfrac{4}{x^2} + \dfrac{27}{y^2} = 39. \end{cases} \quad (f)\ \begin{cases} \dfrac{1}{x + y} - \dfrac{3}{x - y} = -8, \\ \dfrac{2}{x + y} + \dfrac{5}{x - y} = 17. \end{cases}$$

PART II. EQUATIONS IN THREE UNKNOWNS DETERMINANTS OF THIRD ORDER

41. Solution of Three Linear Equations. To solve three linear equations with three variables x, y, z,

$$\text{(1)} \qquad \begin{cases} a_1x + b_1y + c_1z = k_1, \\ a_2x + b_2y + c_2z = k_2, \\ a_3x + b_3y + c_3z = k_3, \end{cases}$$

in a systematic way, we might first eliminate z between the second and third equations (by multiplying the second by c_3, the third by c_2, and subtracting); and then eliminate z between the third and first equations. We should then have two linear equations in x and y, which can be solved as in § 36. This method is long and tedious. But we can find x directly by multiplying the three given equations respectively by

$$b_2c_3 - b_3c_2 = \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}, \; b_3c_1 - b_1c_3 = \begin{vmatrix} b_3 & c_3 \\ b_1 & c_1 \end{vmatrix}, \; b_1c_2 - b_2c_1 = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix},$$

and adding the resulting equations. For it is readily verified that, in the final equation, the coefficients of y and z, viz.

$$b_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} + b_2\begin{vmatrix} b_3 & c_3 \\ b_1 & c_1 \end{vmatrix} + b_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}, \quad c_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} + c_2\begin{vmatrix} b_3 & c_3 \\ b_1 & c_1 \end{vmatrix} + c_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix},$$

are both zero. We find therefore

$$\left[a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} + a_2\begin{vmatrix} b_3 & c_3 \\ b_1 & c_1 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}\right]x$$
$$= \left[k_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} + k_2\begin{vmatrix} b_3 & c_3 \\ b_1 & c_1 \end{vmatrix} + k_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}\right].$$

i.e. if the coefficient of x is $\neq 0$.

$$x = \frac{k_1b_2c_3 - k_1b_3c_2 + k_2b_3c_1 - k_2b_1c_3 + k_3b_1c_2 - k_3b_2c_1}{a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1}.$$

Observe that the numerator is obtained from the denominator by simply replacing every a by the corresponding k.

It can be shown similarly that y is a quotient with the same denominator, and with the numerator obtained from the denominator by replacing every b by the corresponding k; and that z is a quotient with the same denominator and the numerator obtained by replacing every c by the corresponding k.

42. Determinants. The common denominator of x, y, z is usually written in the form

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \tag{2}$$

and is then called a ***determinant of the third order.*** The nine numbers $a_1, b_1, c_1, a_2, b_2, c_2, a_3, b_3, c_3$ are called its ***elements;*** the horizontal lines are called the ***rows,*** the vertical lines the ***columns.*** The diagonal through the first element a_1 is called the ***principal diagonal;*** that through a_3 the ***secondary diagonal.***

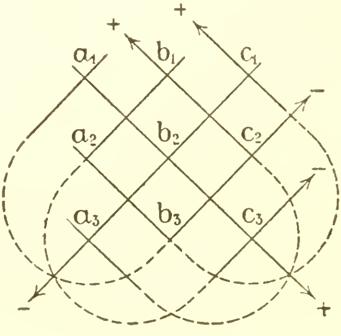
By § 41 we have

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} + a_2 \begin{vmatrix} b_3 & c_3 \\ b_1 & c_1 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

$$= a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1.$$

Thus, a determinant of the third order represents a sum of six *terms, each term being a product of three elements and containing one and only one element from each row and from each column.*

The most convenient method for *expanding* a determinant of the third order, *i.e.* for finding the six terms of which it is the sum, is indicated by the adjoining scheme.



Draw the principal diagonal and the parallels to it, as in the figure; this gives the terms with sign +; then draw the secondary diagonal and the parallels to it; this gives the terms with sign —. (Compare § 14.)

43. General Rule. When three linear equations, like (1), § 41, are given, the determinant (2), § 42, of the coefficients of x, y, z is called the *determinant of the equations.* We can now state the rule for solving the equations (1) when their determinant is different from zero, by the following formulas (compare § 36):

$$x = \frac{\begin{vmatrix} k_1 & b_1 & c_1 \\ k_2 & b_2 & c_2 \\ k_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a_1 & k_1 & c_1 \\ a_2 & k_2 & c_2 \\ a_3 & k_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}, \quad z = \frac{\begin{vmatrix} a_1 & b_1 & k_1 \\ a_2 & b_2 & k_2 \\ a_3 & b_3 & k_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}};$$

i.e. *each of the variables is the quotient of two determinants; the denominator in each case is the determinant of the equations, while the numerator is obtained from this common denominator by replacing the coefficients of the variable by the constant terms.*

It will be shown in solid analytic geometry that any linear equation in x, y, z represents a plane. Hence by solving the three simultaneous equations of § 41 we find the point (or points) common to three planes.

EXERCISES

1. Evaluate the determinants:

(a) $\begin{vmatrix} 1 & 2 & 1 \\ 3 & 1 & 3 \\ 1 & 4 & 1 \end{vmatrix}$. (b) $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$. (c) $\begin{vmatrix} -1 & 1 & 2 \\ 7 & 0 & 3 \\ 6 & -4 & 9 \end{vmatrix}$.

(d) $\begin{vmatrix} 0 & 1 & 3 \\ 4 & 0 & 3 \\ 5 & -1 & 2 \end{vmatrix}$. (e) $\begin{vmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ y & z & 1 \end{vmatrix}$. (f) $\begin{vmatrix} 1 & c & -b \\ -c & 1 & a \\ b & -c & 1 \end{vmatrix}$.

2. Show that

$$\begin{vmatrix} a+b & b & 0 \\ b & b+c & c \\ 0 & c & c+d \end{vmatrix} = abcd\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right).$$

3. Evaluate

$$(a)\ \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \qquad (b)\ \begin{vmatrix} 0 & a & a \\ a & 0 & a \\ a & a & 0 \end{vmatrix}.$$

4. Solve by determinants:

$$(a)\ \begin{cases} 3x+4y+z=5, \\ x-y-z=0, \\ 2x-3y-2z=1. \end{cases} \qquad (b)\ \begin{cases} 3x-4y+7z=8, \\ 2x+3y+6z=-7, \\ x-y=4. \end{cases}$$

$$(c)\ \begin{cases} x+2y-3z=7, \\ x+3y=4, \\ 2x-6y-10z=-8. \end{cases} \qquad (d)\ \begin{cases} x^2+y^2-z^2=-3, \\ 2x^2-y^2+3z^2=62, \\ 5x^2-2y^2-3z^2=-11. \end{cases}$$

$$(e)\ \begin{cases} \dfrac{1}{x^2}-\dfrac{1}{y^2}-\dfrac{2}{z^2}+8=0, \\ \dfrac{2}{x^2}+\dfrac{1}{y^2}-\dfrac{3}{z^2}+9=0, \\ \dfrac{1}{x^2}+\dfrac{4}{y^2}-\dfrac{5}{z^2}+15=0. \end{cases} \qquad (f)\ \begin{cases} \dfrac{2}{x-y}-\dfrac{3}{y-z}=1, \\ \dfrac{4}{z+x}-\dfrac{5}{x-y}=-7, \\ \dfrac{3}{y-z}+\dfrac{2}{z+x}=0. \end{cases}$$

44. Properties of Determinants. — The advantages of using determinants instead of the longer equivalent algebraic expressions of the usual kind will be apparent after studying the principal properties of determinants and the geometrical applications that will follow.

(1) *A determinant is zero whenever all the elements of any row, or all those of any column, are zero.*

This follows from the fact that, in the expanded form (§ 42), every term contains one element from each row and one from each column.

(2) It follows, for the same reason, that *if all elements of any row (or of any column) have a factor in common, this factor can be taken out and placed before the determinant*; thus, *e.g.*,

$$\begin{vmatrix} a_1 & mb_1 & c_1 \\ a_2 & mb_2 & c_2 \\ a_3 & mb_3 & c_3 \end{vmatrix} = m \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

(3) *The value of a determinant is not changed by transposition; i.e.* by making the columns the rows, and *vice versa*, preserving their order. Thus:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix};$$

for, by expanding the determinant on the right we obtain the same six terms, with the same signs, as by expanding the determinant on the left.

(4) *The interchange of any two rows (or of any two columns) reverses the sign, but does not change the absolute value, of the determinant.*

This also follows directly from the expanded form of the determinant (§ 42). For, the interchange of two rows is equivalent to interchanging two subscripts leaving the letters fixed, and this changes every term with the sign + into a term with the sign −, and *vice versa*. The interchange of two columns is equivalent to the interchange of two letters, leaving the subscripts fixed, which has the same effect.

(5) *A determinant in which the elements of any row (column) are equal to the corresponding elements of any other row (column) is zero.*

For, by (4), the sign of the determinant is reversed when any two rows (columns) are interchanged; but the interchange of two *equal* rows (columns) cannot change the value of the determinant. Hence, denoting this value by A, we have in this case $-A = A$, *i.e.* $A = 0$.

EXERCISES

1. Show that

$$\begin{vmatrix} 4 & -3 & 4 \\ 6 & -1 & 5 \\ -10 & 7 & -9 \end{vmatrix} = 2\begin{vmatrix} 2 & 3 & 4 \\ 3 & 1 & 5 \\ 5 & 7 & 9 \end{vmatrix}.$$

2. Evaluate without expanding:

$$(a)\ \begin{vmatrix} 2 & -4 & 3 \\ -7 & 14 & 7 \\ 4 & -8 & 4 \end{vmatrix}. \qquad (b)\ \begin{vmatrix} 7 & 13 & 11 \\ -3 & 0 & 6 \\ 1 & 0 & -2 \end{vmatrix}, \qquad (c)\ \begin{vmatrix} 1000 & 1 & 3 \\ 4 & 1 & 3 \\ 8 & 2 & 6 \end{vmatrix}.$$

3. Without expanding show that

$$(a)\ \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \frac{1}{abc}\begin{vmatrix} 1 & 1 & 1 \\ dbc & eca & fab \\ gbc & hca & iab \end{vmatrix}; \qquad (b)\ \begin{vmatrix} bc & a & 1 \\ ca & b & 1 \\ ab & c & 1 \end{vmatrix} = \begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix}.$$

45. Expansion by Minors. The general type of a determinant of the third order is often written in the form

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix},$$

so that the first subscript indicates the row, the second the column in which the element stands. Any one of the nine elements is denoted by a_{ik}.

If in a determinant of the third order, both the row and the column in which any particular element a_{ik} stands be struck out, the remaining elements form a determinant of the second order, which is called the ***minor*** of the element a_{ik}. Thus the minor of a_{23} is

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}.$$

By § 42 we have

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{21}\begin{vmatrix} a_{32} & a_{33} \\ a_{12} & a_{13} \end{vmatrix} + a_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix};$$

the right-hand member is called the *expansion of the determinant by minors of the* (elements of the) *first column*. It should be noticed, however, that, while the coefficients of a_{11} and a_{31} in this expansion are the minors of these elements, the coefficient of a_{21} is *minus* the minor of a_{21}.

The determinant can also be expanded by minors of the second column:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{12}\begin{vmatrix} a_{31} & a_{33} \\ a_{21} & a_{23} \end{vmatrix} + a_{22}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + a_{32}\begin{vmatrix} a_{21} & a_{23} \\ a_{11} & a_{13} \end{vmatrix};$$

here the coefficients of a_{12} and a_{32} are *minus* the minors of these elements while the coefficient of a_{22} is the minor of a_{22} itself. This expansion follows from the previous one because the value of the determinant merely changes sign when the first and second columns are interchanged.

Let the student write out the similar development in terms of minors of the third column.

As the value of the determinant is not changed by transposition (§ 44 (3)), the determinant may also be expanded by minors of the elements of any *row*.

46. Cofactors. To sum up these results briefly, let us denote by A the value of the determinant itself, and by A_{ik} the value of the minor of the element a_{ik}, *multiplied by* $(-1)^{i+k}$, *i.e.* the so-called ***cofactor*** of a_{ik}. We then have:

$$\begin{aligned} A &= a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31}, \\ &= a_{12}A_{12} + a_{22}A_{22} + a_{32}A_{32}, \\ &= a_{13}A_{13} + a_{23}A_{23} + a_{33}A_{33}, \end{aligned}$$

and similarly for the expansion by minors, or rather cofactors, of any row.

At the same time it should be noted that if we add the elements of any column (row) each multiplied by the cofactors of any *other* column (row), the result is always zero. Thus it is readily verified that

$$\begin{aligned} a_{11}A_{12} + a_{21}A_{22} + a_{31}A_{32} &= 0, \\ a_{11}A_{13} + a_{21}A_{23} + a_{31}A_{33} &= 0, \\ a_{12}A_{11} + a_{22}A_{21} + a_{32}A_{31} &= 0, \end{aligned}$$

etc. This property was used in § 41.

47. Sum of Two Determinants. *If all the elements of any column (or row) are sums, the determinant can be resolved into a sum of determinants.* Thus, if all elements of the first column are sums of two terms, we find, expanding by minors of the first column:

$$\begin{aligned} \begin{vmatrix} a_1+m_1 & b_1 & c_1 \\ a_2+m_2 & b_2 & c_2 \\ a_3+m_3 & b_3 & c_3 \end{vmatrix} &= (a_1+m_1)\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} + (a_2+m_2)\begin{vmatrix} b_3 & c_3 \\ b_1 & c_1 \end{vmatrix} + (a_3+m_3)\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} \\ &= a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} + a_2\begin{vmatrix} b_3 & c_3 \\ b_1 & c_1 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} \\ &\quad + m_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} + m_2\begin{vmatrix} b_3 & c_3 \\ b_1 & c_1 \end{vmatrix} + m_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} \\ &= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} m_1 & b_1 & c_1 \\ m_2 & b_2 & c_2 \\ m_3 & b_3 & c_3 \end{vmatrix}. \end{aligned}$$

Let the student show, by interchanging rows and columns, that the same property holds for rows.

As any row (column) can be made the first by interchanging it with the first and changing the sign of the determinant, this decomposition into the sum of two determinants is possible whenever every element of *any one* row or column is a sum.

As a particular case we have

$$\begin{vmatrix} a_1+b_1 & b_1 & c_1 \\ a_2+b_2 & b_2 & c_2 \\ a_3+b_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} b_1 & b_1 & c_1 \\ b_2 & b_2 & c_2 \\ b_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},$$

since the second determinant, which has two equal columns, is zero by (5), § 44. We conclude that *the value of a determinant is not changed by adding to each element of any row (column) the corresponding element of any other row (column).* Indeed, owing to (2), § 44, we can add to each element of any row (column) the corresponding element of any other row (column) *multiplied by one and the same factor.* This property is of great help in reducing a given determinant to a more simple form and evaluating it.

In the case of a numerical determinant, it is often best after taking out the common factors from any row or column to reduce two elements of some row or column to zero, by addition or subtraction. Thus, taking out the factors 2 from the third column and 3 from the second row, we have

$$A = \begin{vmatrix} 2 & 3 & -14 \\ 3 & 18 & -12 \\ -4 & 8 & 18 \end{vmatrix} = 6 \begin{vmatrix} 2 & 3 & -7 \\ 1 & 6 & -2 \\ -4 & 8 & 9 \end{vmatrix};$$

subtracting twice the second row from the first and adding 4 times the second row to the third, we find

$$A = 6 \begin{vmatrix} 0 & -9 & -3 \\ 1 & 6 & -2 \\ 0 & 32 & 1 \end{vmatrix} = -6 \begin{vmatrix} -9 & -3 \\ 32 & 1 \end{vmatrix} = 18 \begin{vmatrix} 3 & 1 \\ 32 & 1 \end{vmatrix} = -522.$$

EXERCISES

1. Evaluate the determinants:

$$(a)\ \begin{vmatrix} 1 & 3 & 7 \\ 3 & 5 & 9 \\ 4 & 8 & 16 \end{vmatrix}, \qquad (b)\ \begin{vmatrix} 27 & 26 & 27 \\ 31 & 33 & 36 \\ 43 & 44 & 45 \end{vmatrix}, \qquad (c)\ \begin{vmatrix} 17 & 34 & 51 \\ 28 & 72 & 38 \\ 39 & 65 & 52 \end{vmatrix},$$

$$(d)\ \begin{vmatrix} 6 & 33 & 9 \\ 14 & 21 & 35 \\ 26 & 39 & 42 \end{vmatrix}, \qquad (e)\ \begin{vmatrix} 7 & 17 & 29 \\ 11 & 19 & 31 \\ 13 & 23 & 37 \end{vmatrix}, \qquad (f)\ \begin{vmatrix} 2 & -3 & 40 \\ 5 & 7 & -10 \\ 3 & -2 & 60 \end{vmatrix}.$$

2. Show that

$$(a)\ \begin{vmatrix} b+c & a & 1 \\ c+a & b & 1 \\ a+b & c & 1 \end{vmatrix} = 0; \qquad (b)\ \begin{vmatrix} 1 & a^2-d^2 & a^3-d^3 \\ 1 & b^2-d^2 & b^3-d^3 \\ 1 & c^2-d^2 & c^3-d^3 \end{vmatrix} = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix};$$

$$(c)\quad \begin{vmatrix} b_1+c_1 & c_1+a_1 & a_1+b_1 \\ b_2+c_2 & c_2+a_2 & a_2+b_2 \\ b_3+c_3 & c_3+a_3 & a_3+b_3 \end{vmatrix} = 2\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix};$$

$$(d)\quad \begin{vmatrix} a+b & a+4b & a+7b \\ a+2b & a+5b & a+8b \\ a+3b & a+6b & a+9b \end{vmatrix} = 0.$$

48. Elimination. Three *homogeneous* linear equations,

$$(3)\qquad \begin{cases} a_1x+b_1y+c_1z=0, \\ a_2x+b_2y+c_2z=0, \\ a_3x+b_3y+c_3z=0, \end{cases}$$

are obviously satisfied by $x=0$, $y=0$, $z=0$. Can they have other solutions?

Solving the equations by the method of § 43, and denoting the determinant of the equations for the sake of brevity by A, we find since $k_1=0$, $k_2=0$, $k_3=0$:

$$Ax=0,\quad Ay=0,\quad Az=0.$$

Hence, if x, y, z are not all three zero, we must have $A=0$. *Three homogeneous linear equations can therefore have solutions that are not all zero only if the determinant of the equations is equal to zero.*

If x, for instance, is different from zero, we can divide each of the three equations by x and then eliminate y/x and z/x between the three equations. The result is $A=0$, *i.e.*

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$$

Thus, *the result of eliminating the three variables between three homogeneous linear equations is the determinant of the equations equated to zero.* (Compare § 40.)

Solving the first and second equations for y/x, z/x, we obtain

$$\frac{x}{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}} = \frac{y}{\begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}} = \frac{z}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}},$$

provided the denominators are all different from zero.

With the notation of § 46, this can be written $x:y:z = A_{31}:A_{32}:A_{33}$. If we solve the third and first or second and third equations for y/x, z/x, we find, respectively, $x:y:z = A_{21}:A_{22}:A_{23}$, or $x:y:z = A_{11}:A_{12}:A_{13}$. Hence, whenever $A = 0$, we can find the *ratios* of the variables unless *all* the minors of A are zero.

49. Geometric Applications. The *equation of a line through two points* $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ can be found as follows. The equation of *any* line must be of the form (§ 30)

$$Ax + By + C = 0. \tag{4}$$

The question is to determine the coefficients A, B, C, so that the line shall pass through the points P_1 and P_2. If the line is to pass through the point P_1, the equation must be satisfied by the coordinates x_1, y_1 of this point, *i.e.* we must have

$$Ax_1 + By_1 + C = 0;$$

this is the first condition to be satisfied by the coefficients. In the same way we find the second condition

$$Ax_2 + By_2 + C = 0.$$

We might calculate from these two conditions the values of A/C and B/C and then substitute these values in the first equation. But as this means merely eliminating A, B, C between the three equations, we can obtain the result directly (§ 48) by equating to zero the determinant of the coefficients of A, B, C.

Thus the ***equation of the line through two points*** P_1, P_2 is:

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0. \tag{5}$$

Observe that this equation is evidently satisfied if x, y are replaced either by x_1, y_1 or by x_2, y_2 (see (5), § 44).

50. Area of a Triangle. The area A of a triangle $P_1P_2P_3$ in terms of the coordinates of its vertices $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, $P_3(x_3, y_3)$ is:

$$A = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix};$$

for, upon expanding this determinant, we find the value given before in § 14.

It will now be seen that the determinant equation (5) of the line through two points given in § 49 merely expresses the fact that any point (x, y) of the line forms with the given points (x_1, y_1) and (x_2, y_2) a triangle whose area is zero.

EXERCISES

1. Write down the equation of the line through $(2, 3)$, $(-2, \frac{1}{3})$; expand the determinant by minors of the first row; determine the slope and the intercepts; sketch the line.

2. Find the equation of the line through the points: $(3, -4)$ and $(0, 2)$; $(0, b)$ and $(a, 0)$; $(0, 0)$ and $(2, 1)$.

3. Find the area of the triangle whose vertices are the points $(1, 1)$, $(2, -3)$, $(5, -8)$.

4. Find the area of the quadrilateral whose vertices are the points $(3, -2)$, $(4, -5)$, $(-3, 1)$, $(0, 0)$.

5. If the base of a triangle joins the points $(-1, 2)$ and $(4, 3)$, on what line does the vertex lie if the area of the triangle is equal to 6?

6. Find the coordinates of the common vertex of the two triangles of equal area 3, whose bases join the points $(3, 5)$, $(6, -8)$ and $(3, -1)$, $(2, 2)$, respectively.

7. Show that the area of any triangle is four times the area of the triangle formed by joining the midpoints of its sides.

8. Show that the sum of the areas of the triangles whose vertices are (a, d), $(2b, e)$, $(5c, f)$, and $(3a, d)$, $(4b, e)$, $(3c, f)$ is given by the determinant

$$\begin{vmatrix} 2a & d & 1 \\ 3b & e & 1 \\ 4c & f & 1 \end{vmatrix}.$$

9. Show that the lines joining the midpoints of the sides of any triangle divide the triangle into four equal triangles.

10. Show that the condition that the three lines $Ax + By + C = 0$, $A'x + B'y + C' = 0$, $A''x + B''y + C'' = 0$ meet at a point is

$$\begin{vmatrix} A & B & C \\ A' & B' & C' \\ A'' & B'' & C'' \end{vmatrix} = 0.$$

11. Show that the straight lines $3x + y - 1 = 0$, $x - 3y + 13 = 0$, $2x - y + 6 = 0$ have a common point.

12. For what values of s do the following lines meet in a point:

$$4x - 6y + s = 0,\quad sx - 36y = 0,\quad x + y - 1 = 0\ ?$$

13. Show that the altitudes of any triangle meet in a point.

14. Show that the medians of any triangle meet in a point.

15. Show that the line through the origin perpendicular to the line through the points $(a, 0)$ and $(0, b)$ meets the lines through the points $(a, 0)$, $(-b, b)$ and $(0, b)$, $(a, -a)$ in a common point.

16. Show that the distance of the point $P_1(x_1, y_1)$ from the line joining the points $P_2(x_2, y_2)$ and $P_3(x_3, y_3)$ is

$$h = \frac{\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}}{\sqrt{(x_3 - x_2)^2 + (y_3 - y_2)^2}}.$$

CHAPTER IV

RELATIONS BETWEEN TWO OR MORE LINES

51. Angle between Two Lines. We shall understand by the angle $(l, l') = \theta$ between two lines l and l' the least angle through which l must be turned counterclockwise about the point of intersection to come to coincidence with l'. This angle θ is equal to the difference of the slope angles α, α' (Fig. 27) of the two lines. Thus, if $\alpha' > \alpha$, we have $\theta = \alpha' - \alpha$, since α' is the exterior angle of a triangle, two of whose interior angles are α and θ. It follows that

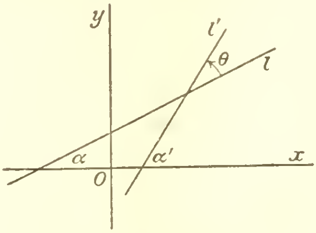


Fig. 27

$$\tan\theta = \tan(\alpha' - \alpha) = \frac{\tan\alpha' - \tan\alpha}{1 + \tan\alpha\tan\alpha'}. \tag{1}$$

If the equations of l and l' are

$$y = mx + b, \quad y = m'x + b',$$

respectively, we have $\tan\alpha = m$, $\tan\alpha' = m'$; hence

$$\tan\theta = \frac{m' - m}{1 + mm'}. \tag{2}$$

If the equations of l and l' are

$$Ax + By + C = 0,$$
$$A'x + B'y + C' = 0,$$

respectively, we have $\tan\alpha = -A/B$, $\tan\alpha' = -A'/B'$; hence

$$\tan\theta = \frac{AB' - A'B}{AA' + BB'}. \tag{3}$$

52. It follows, in particular, that the two lines l and l', § 51, are ***parallel*** if and only if

$$m' = m, \quad \text{or } AB' - A'B = 0;$$

and they are ***perpendicular*** to each other if and only if

$$m' = -\frac{1}{m}, \text{ or } AA' + BB' = 0.$$

(Compare §§ 27, 31.) Hence, to write down the equation of a line *parallel* to a given line, replace the constant term by an arbitrary constant: to write down the equation of a line *perpendicular* to a given line, interchange the coefficients of x and y, changing the sign of one of them, and replace the constant term by an arbitrary constant.

EXERCISES

1. Determine whether the following pairs of lines are parallel or perpendicular: $3x + 2y - 6 = 0$, $2x - 3y + 4 = 0$; $5x + 3y - 6 = 0$, $10x + 6y + 2 = 0$; $2x + 5y - 14 = 0$, $8x - 3y + 6 = 0$.

2. Find the point of intersection of the line $5x + 8y + 17 = 0$ with its perpendicular through the origin.

3. Find the point of intersection of the lines through the points $(6, -2)$ and $(0, 2)$, and $(4, 5)$ and $(-1, -4)$.

4. Find the perpendicular bisector of the line-segment joining the point $(3, 4)$ to the point of intersection of the lines $2x - y + 1 = 0$ and $3x + y - 16 = 0$.

5. Find the lines through the point of intersection of the lines $5x - y = 0$, $x + 7y - 9 = 0$ and perpendicular to them.

6. Find the area of the triangle formed by the lines $3x + 4y = 8$, $6x - 5y = 30$, and $x = 0$.

7. Find the area of the triangle formed by the lines $x + y - 1 = 0$, $2x + y + 5 = 0$, and $x - 2y - 10 = 0$.

8. Find the point of intersection of the lines

$$(a) \quad \frac{x}{a} + \frac{y}{b} = 1, \quad \frac{x}{b} + \frac{y}{a} = 1.$$

$$(b) \quad \frac{x}{a} + \frac{y}{b} = 1, \quad y = mx + b.$$

9. Find the area of the triangle formed by the lines $y = m_1x + b_1$, $y = m_2x + b_2$ and the axis Ox.

10. The vertices of a triangle are $(5, -4)$, $(-3, 2)$, $(7, 6)$. Find the equations of the medians and their point of intersection.

11. Find the angle between the lines $4x - 3y - 6 = 0$ and $x - 7y + 6 = 0$.

12. Find the tangent of the angle between the lines (*a*) $4x - 3y + 6 = 0$ and $9x + 2y - 8 = 0$; (*b*) $3x + 6y - 11 = 0$ and $x + 2y - 3 = 0$.

13. Find the two lines through the point (6, 10) inclined at 45° to the line $3x - 2y - 12 = 0$.

14. Find the lines through the point $(-3, 7)$ such that the tangent of the angle between each of these lines and the line $6x - 2y + 11 = 0$ is $\frac{7}{3}$.

15. Show that the angle between the lines $Ax + By + C = 0$ and $(A + B)x - (A - B)y + D = 0$ is 45°.

16. Find the lines which make an angle of 45° with the line $4x - 7y + 6 = 0$ and bisect the portion of it intercepted by the axes.

17. The hypotenuse of an isosceles right-angled triangle lies on the line $3x - 6y - 17 = 0$. The origin is one vertex; what are the others?

53. Polar Equation of Line. The position of a line in the plane is fully determined by the length $p = ON$ (Fig. 28) of the perpendicular let fall from the origin on the line and the angle $\beta = xON$ made by this perpendicular with the axis Ox.

Then p and β are evidently the *polar* coordinates of the point N (§ 16). Let P be any point of the line and $OP = r$, $xOP = \phi$ its polar coordinates. As the projection of OP on the perpendicular ON is equal to ON, and the angle $NOP = \phi - \beta$, we have

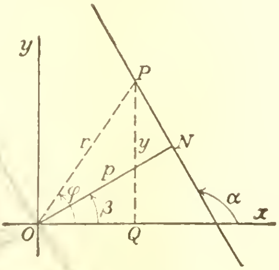


FIG. 28

$$r \cos(\phi - \beta) = p. \tag{4}$$

This is the *equation of the line NP in polar coordinates.*

54. Normal Form. The last equation can be transformed to Cartesian coordinates by expanding the cosine:

$$r \cos \phi \cos \beta + r \sin \phi \sin \beta = p$$

and observing (§ 17) that $r \cos \phi = x$, $r \sin \phi = y$; the equation then becomes

$$x \cos \beta + y \sin \beta = p. \tag{5}$$

This equation, which is called the ***normal form*** of the equation of the line, can be read off directly from the figure; it means that the sum of the projections of x and y on the perpendicular to the line is equal to the projection of r (§ 20).

Observe that in the normal form (5) the number p is always positive, being the distance of the line from the origin, or the radius vector of the point N. Hence $x \cos \beta + y \sin \beta$ is always positive; this also appears by considering that $x \cos \beta + y \sin \beta$ is the projection of the radius vector OP on ON, and that this radius vector makes with ON an angle that cannot be greater than a right angle.

The angle $\beta = xON$ is, as a polar angle (§ 16), always understood to be the angle through which the axis Ox must be turned counterclockwise about the origin to make it coincide with ON; it can therefore have any value from 0 to 2π. By drawing the parallel to the line NP through the origin it is readily seen that, if α is the slope angle of the line NP, we have

$$\beta = \alpha + \tfrac{1}{2}\pi \quad \text{or} \quad \beta = \alpha + \tfrac{3}{2}\pi$$

according as the line lies on one side of the origin or the other, angles differing by 2π being regarded as equivalent. Thus, in Fig. 28, $\alpha = 120°$, $\beta = \alpha + \frac{3}{2}\pi = 120° + 270° = 390°$, which is equivalent to $30°$. For a parallel on the opposite side of the origin we should have $\beta = \alpha + \frac{1}{2}\pi = 120° + 90° = 210°$.

55. Reduction to Normal Form. The equation

$$Ax+By+C=0$$

is in general not of the form (5), since in the latter equation the coefficients of x and y, being the cosine and sine of an angle, have the property that the sum of their squares is equal to 1, while in the former equation the sum of the squares of A and B is in general not equal to 1. But the general equation

$$Ax+By+C=0$$

can be reduced to the normal form (5) by multiplying it by a factor k properly chosen; we know (§ 30) that the equation

$$kAx+kBy+kC=0$$

represents the same line as does the equation $Ax+By+C=0$. Now if we select k so that

$$kA=\cos\beta,\quad kB=\sin\beta,\quad kC=-p,$$

the equation $Ax+By+C=0$ reduces to the normal form $x\cos\beta+y\sin\beta-p=0$. The first two conditions give

$$k^2A^2+k^2B^2=\cos^2\beta+\sin^2\beta=1,$$

whence
$$k=\pm\frac{1}{\sqrt{A^2+B^2}}.$$

Since the right-hand member p in the normal form (5) is positive, the sign of the square root must be selected so that kC becomes negative. We have therefore the rule:

To reduce the general equation $Ax+By+C=0$ *to the normal form*

$$x\cos\beta+y\sin\beta-p=0,$$

divide by $-\sqrt{A^2+B^2}$ *when* C *is positive and by* $+\sqrt{A^2+B^2}$ *when* C *is negative.*

Then the coefficients of x and y will be $\cos\beta$, $\sin\beta$, respectively, and the constant term will be the distance p of the line from the origin.

Thus, to reduce $3x + 2y + 5 = 0$ to the normal form, divide by $-\sqrt{3^2 + 2^2} = -\sqrt{13}$; this gives

$$\cos\beta = -\frac{3}{\sqrt{13}},\ \sin\beta = -\frac{2}{\sqrt{13}},\ -p = -\frac{5}{\sqrt{13}};$$

i.e. the normal form is

$$-\frac{3}{\sqrt{13}}x - \frac{2}{\sqrt{13}}y = \frac{5}{\sqrt{13}}.$$

The perpendicular to the line from the origin has the length $5/\sqrt{13}$; and as both $\cos\beta$ and $\sin\beta$ are negative, this perpendicular lies in the third quadrant. Draw the line.

Reduce the equation $3x + 2y - 5 = 0$ to the normal form.

56. Distance of a Point from a Line. If, in Fig. 28, we take instead of a point P *on* the line any point P_1 (x_1, y_1) *not* on the line (Fig. 29), the expression $x_1\cos\beta + y_1\sin\beta$ is still the projection on ON (produced if necessary) of the radius vector OP_1. But this projection OS differs from the normal $ON = p$ to the line. The figure shows that the difference

$$x_1\cos\beta + y_1\sin\beta - p = OS - ON = NS$$

is equal to the distance N_1P_1 of the point P_1 *from* the line.

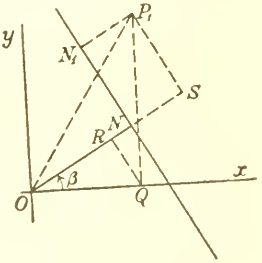


Fig. 29

Thus, to find the distance of any point P_1 (x_1, y_1) from a line whose equation is given in the normal form

$$x\cos\beta + y\sin\beta - p = 0,$$

it suffices to substitute in the left-hand member of this equation for x, y the coordinates x_1, y_1 of the point P_1. The expression

$$x_1\cos\beta + y_1\sin\beta - p$$

then represents *the distance of P_1 from the line.*

If this expression is negative, the point P_1 lies on the same side of the line as does the origin; if it is positive, the point

P_1 lies on the opposite side of the line. Any line thus divides the plane into two regions which we may call the positive and negative regions; that in which the origin lies is the negative region.

To find the distance of a point P_1 (x_1, y_1) from a line given in the general form

$$Ax + By + C = 0,$$

we have only to reduce the equation to the normal form (§ 55) and then apply the rule given above. Thus the distance is

$$\frac{Ax_1 + By_1 + C}{-\sqrt{A^2 + B^2}} \quad \text{or} \quad \frac{Ax_1 + By_1 + C}{\sqrt{A^2 + B^2}},$$

according as C is positive or negative.

57. Bisector of an Angle. To find the ***bisectors*** of the angles between two lines given in the normal form

$$x \cos \beta + y \sin \beta - p = 0,$$
$$x \cos \beta' + y \sin \beta' - p' = 0,$$

observe that for any point on either bisector its distances from the two lines must be equal in absolute value. Hence the equations of the bisectors are

$$x \cos \beta + y \sin \beta - p = \pm (x \cos \beta' + y \sin \beta' - p').$$

To distinguish the two bisectors, observe that for the bisector of that pair of vertical angles which contains the origin (Fig. 30) the perpendicular distances are, in one angle both positive, in the other both negative; hence the plus sign gives this bisector.

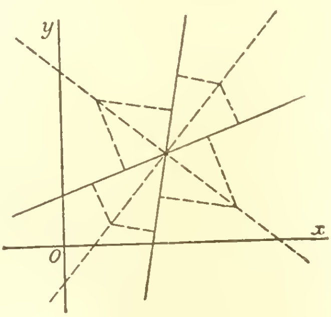


FIG. 30

If the equations of the lines are given in the general form

$$Ax + By + C = 0, \quad A'x + B'y + C' = 0,$$

first reduce the equations to the normal form, and then apply the previous rule.

EXERCISES

1. Draw the lines represented by the following equations :

(*a*) $r \cos (\phi - \frac{1}{6}\pi) = 6.$	(*e*) $r \cos (\phi + \frac{3}{2}\pi) = 3.$
(*b*) $r \cos (\phi - \pi) = 4.$	(*f*) $r \sin (\phi - \frac{1}{3}\pi) = 8.$
(*c*) $r \cos \phi = 10.$	(*g*) $r \sin (\phi + \frac{3}{4}\pi) = 7.$
(*d*) $r \sin \phi = 5.$	(*h*) $r \cos (\phi - \frac{2}{3}\pi) = 6.$

2. In polar coordinates, find the equations of the lines : (*a*) parallel to and at the distance 5 from the polar axis (above and below) ; (*b*) perpendicular to the polar axis and at the distance 4 from the pole (to the right and left) ; (*c*) inclined at an angle of $\frac{1}{3}\pi$ to the polar axis and at the distance 12 from the pole.

3. Express in polar coordinates the sides of the rectangle $OABC$ if $OA = 6$ and $AB = 9$, OA being taken as polar axis.

4. What lines are represented by (5) when p is constant, while β varies from zero to 2π ? What lines when p varies while β remains constant ?

5. The perpendicular from the origin to a line is 5 units in length and makes an angle $\tan^{-1}\frac{5}{12}$ with the axis Ox. Find the equation of the line.

6. Reduce the equations of Ex. 8, p. 34, to the normal form (5).

7. Find the equations of the lines whose slope angle is 150° and which are at the distance 4 from the origin.

8. What is the equation of the line through the point $(-3, 5)$ whose perpendicular from the origin makes an angle of 120° with the axis Ox?

9. For the line $7x - 24y - 20 = 0$ find the intercepts, slope, length of perpendicular from the origin and the sine and cosine of the angle which this perpendicular makes with the axis Ox.

10. Find by means of $\sin \beta$ and $\cos \beta$ the quadrants crossed by the line $4x - 5y = 8$.

11. Put the following equations in the form (5) and thus find p, $\sin \beta$, $\cos \beta$:

(*a*) $y = mx + b.$ (*b*) $\frac{x}{a} + \frac{y}{b} = 1.$ (*c*) $3x = 4y.$

12. Is the point $(3, -4)$ on the positive or negative side of the line through the points $(-5, 2)$ and $(4, 7)$?

13. Is the point $(-1, -\frac{3}{2})$ on the positive or negative side of the line $4x - 9y - 8 = 0$?

14. Find by means of an altitude and a side the area of the triangle formed by the lines $3x + 2y + 10 = 0$, $4x - 3y + 16 = 0$, $2x + y - 4 = 0$. Check the result with another altitude and side.

15. Find the distance between the parallel lines (*a*) $3x - 5y - 4 = 0$ and $6x - 10y + 7 = 0$; (*b*) $5x + 7y + 9 = 0$ and $15x + 21y - 3 = 0$.

16. What is the length of the perpendicular from the origin to the line through the point $(-5, -4)$ whose slope angle is 60°?

17. What are the equations of the lines whose distances from the origin are 6 units each and whose slopes are $\frac{5}{3}$?

18. Find the points on the axis Ox whose perpendicular distances from the line $24x - 7y - 16 = 0$ are ± 5.

19. Find the point equidistant from the points $(4, -3)$ and $(-2, 1)$, and at the distance 4 from the line $3x - 4y - 5 = 0$.

20. Find the line parallel to $12x - 5y - 6 = 0$ and at the same distance from the origin; farther from the origin by a distance 3.

21. Find the two lines through the point $(1, \frac{20}{3})$ such that the perpendiculars let fall from the point $(6, 5)$ are of length 5.

22. Find the line perpendicular to $4x - 7y - 10 = 0$ which crosses the axis Ox at a distance 6 from the point $(-2, 0)$.

23. Find the bisectors of the angles between the lines: (*a*) $x - y - 4 = 0$ and $3x + 3y + 7 = 0$; (*b*) $5x - 12y - 16 = 0$ and $24x + 7y + 60 = 0$.

24. Find the bisectors of the angles of the triangle formed by the lines $5x + 12y + 20 = 0$, $4x - 3y - 6 = 0$, $3x - 4y + 5 = 0$ and the center of the circle inscribed in the triangle.

25. Find the bisector of that angle between the lines $3x - \sqrt{3}y + 10 = 0$, $\sqrt{2}x + y - 6 = 0$ in which the origin lies.

26. If two lines are given in the normal form, what is represented by their sum and what by their difference?

27. Show that the angle between the lines $x + y = 0$ and $x - y = 0$ is 90° whether the axes are rectangular or oblique.

58. Pencils of Lines. All lines through one and the same point are said to form a ***pencil;*** the point is called the ***center*** of the pencil. If

$$(6)\qquad \begin{cases} Ax + By + C = 0, \\ A'x + B'y + C' = 0 \end{cases}$$

are any two different lines of a pencil, the equation

$$(7)\qquad Ax + By + C + k(A'x + B'y + C') = 0,$$

where k is any constant, represents a line of the pencil. For, the equation (7) is of the first degree in x and y, and the coefficients of x and y cannot be both zero, since this would mean that the lines (6) are parallel. Moreover, the line (7) passes through the center of the pencil (6) because the coordinates of the point that satisfies each of the equations (6) also satisfy the equation (7).

All lines parallel to the same direction are said to form a *pencil of parallels.* It is readily seen that if the lines (6) are parallel, the equation (7) represents a line parallel to them.

EXERCISES

1. Find the line: (a) through the point of intersection of the lines $4x - 7y + 5 = 0$, $6x + 11y - 7 = 0$ and the origin; (b) through the point of intersection of the lines $4x - 2y - 3 = 0$, $x + y - 5 = 0$ and the point $(-2, 3)$; (c) through the point of intersection of the lines $4x - 5y + 6 = 0$, $y - x - 3 = 0$, of slope 3; (d) through the intersection of $5x - 6y + 10 = 0$, $2x + 3y - 12 = 0$, perpendicular to $4y + x = 0$.

2. Find the line of the pencil $x - 5 = 0$, $y + 2 = 0$ that is inclined to the axis Ox at 30°.

3. Determine the constant b of the line $y = 3x + b$ so that this line shall belong to the pencil $3x - 4y + 6 = 0$, $x = 5$.

4. Find the line joining the centers of the pencils $x - 3y = 12$, $5x - 2y = 1$ and $x + y = 6$, $4x - 5y = 3$.

5. Find the line of the pencil $4x - 5y - 12 = 0$, $3x + 2y - 16 = 0$ that makes equal intercepts on the axes.

59. Non-linear Equations representing Lines. When two lines are given, say

$$Ax + By + C = 0,$$
$$A'x + B'y + C' = 0,$$

then the equation

$$(Ax + By + C)(A'x + B'y + C') = 0,$$

obtained by multiplying the left-hand members (the right-hand members being reduced to zero) is satisfied by all the points of the first given line as well as all the points of the second given line, and by no other points.

The product equation which is of the second degree is therefore said to represent the two given lines. Similarly, by equating to zero the product of the left-hand members of the equations of three or more straight lines (whose right-hand members are zero) we find a single equation representing all these lines. An equation of the nth degree *may* therefore represent n straight lines, viz. when its left-hand member (the right-hand member being zero) can be resolved into n linear factors, with real coefficients.

EXERCISES

1. Find the common equation of the two axes of coordinates.

2. Show that n lines through the origin are represented by a *homogeneous* equation (*i.e.* one in which all terms are of the same degree in x and y) of the nth degree.

3. Draw the lines represented by the following equations:

(*a*) $(x - a)(y - b) = 0.$
(*b*) $3x^2 - xy - 4y^2 = 0.$
(*c*) $x^2 - 9y^2 = 0.$
(*d*) $ax^2 + by^2 = 0.$
(*e*) $x^2 - x - 12 = 0.$
(*f*) $xy - ax = 0.$
(*g*) $y^3 - 5y^2 + 6y = 0.$
(*h*) $x^3y - xy = 0.$
(*i*) $y^3 - 6xy^2 + 11x^2y - 6x^3 = 0.$

4. What relation must hold between a, h, b, if the lines represented by $ax^2 + 2hxy + by^2 = 0$ are to be real and distinct, coincident, imaginary?

MISCELLANEOUS EXERCISES

1. Find the angle between the lines represented by the equation $ax^2 + 2hxy + by^2 = 0$. What is the condition for these lines to be perpendicular? coincident?

2. Reduce the general equation $Ax + By + C = 0$ to the normal form $x \cos \beta + y \sin \beta = p$ by considering that, if both equations represent the same line, the intercepts must be the same.

3. Find the line through (x_1, y_1) making equal intercepts on the axes.

4. Find the area of the triangle formed by the lines $y = m_1x + b_1$, $y = m_2x + b_2$, $y = b$.

5. What does the equation $\phi =$ const. represent in polar coordinates?

6. Find the polar equation of the line through $(6, \pi)$ and $(4, \frac{1}{2}\pi)$.

7. Derive the determinant expression for the area of a triangle (§ 14) by multiplying one side by half the altitude.

8. The weights w, W being suspended at distances d, D, respectively, from the fulcrum of a lever, we have by the *law of the lever* $WD = wd$. If the weights are shifted along the lever, then to every value of d corresponds a definite value of D; *i.e.* D is a function of d. Represent this function graphically; interpret the part of the line in the third quadrant.

9. A train, after leaving the station A, attains in the first 6 minutes, $1\frac{1}{2}$ miles from A, the speed of 30 miles per hour with which it goes on. How far from A will it be 50 minutes after starting? (Compare Example 4, § 29.) Illustrate graphically, taking s in miles, t in minutes.

10. A train leaves Detroit at 8 hr. 25 m. A.M. and reaches Chicago at 4 hr. 5 m. P.M.; another train leaves Chicago at 10 hr. 30 m. A.M. and arrives in Detroit at 5 hr. 30 m. P.M. The distance is 284 miles. Regarding the motion as uniform and neglecting the stops, find graphically and analytically where and when the trains meet. If the scale of distances (in miles) be taken 1/20 of the scale of times (in hours), how can the velocities be found from the slopes?

11. A stone is dropped from a balloon ascending vertically at the rate of 24 ft./sec.; express the velocity as a function of the time (Example 5, § 29). What is the velocity after 4 sec.?

12. How long will a ball rise if thrown vertically upward with an initial velocity of 100 ft./sec.?

CHAPTER V

PERMUTATIONS AND COMBINATIONS. DETERMINANTS OF ANY ORDER

60. Introduction. In using determinants of the second and third order we have seen how advantageous it is to *arrange* conveniently the symbols of an algebraic expression. Before proceeding to the study of the general determinant of the nth order, we must discuss very briefly that branch of algebra which is concerned with the theory of arrangements and changes of arrangement (permutations and combinations). The results are important not only for determinants, but are used very often, even in the common affairs of life; they form, moreover, the basis of the theory of "choice and chance," or of probabilities.

The "things" to be arranged or combined need not be numbers (as they are in a determinant), but may be any whatever, provided they are, and remain, clearly distinguishable from each other; we shall call them ***elements*** and designate them by letters a, b, c, etc.

61. Permutations. Any two elements, a and b, can obviously be arranged in a row in 2 ways:

$$ab, \quad ba.$$

Three elements a, b, c can be arranged in a row in 6, and only 6, ways:

$$\begin{matrix} abc & bac & cab \\ acb & bca & cba \end{matrix}$$

The question arises: in how many ways can n elements be arranged in a row?

Any arrangement of n elements in a row is called a ***permutation.*** It is found by trial that the number of permutations of n elements increases very rapidly with their number n. Thus for 4 elements it is 24, for 5 elements 120. It will be shown that *for n elements the number of permutations is* $1 \cdot 2 \cdot 3 \cdots n$. This expression, the product of the first n positive integers, is briefly designated by $n\,!$, or $\underline{n|}$, and is called ***factorial*** n:

$$n\,! = 1 \cdot 2 \cdot 3 \cdots n.$$

If we denote by P_n the number of permutations of n elements our proposition is

$$P_n = n\,!$$

62. Mathematical Induction. The proof of the proposition that $P_n = n\,!$ is obtained by an important method of reasoning called ***mathematical induction.***

By actual trial we can readily find that $P_1 = 1$, $P_2 = 2$, $P_3 = 6$, and with sufficient patience we might even ascertain that $P_6 = 720$. But to prove the general proposition that $P_n = n\,!$ we must look into the method by which in the particular cases we make sure that we have found all the possible permutations. This method consists in proceeding step by step:

Seeing that 2 elements have 2 permutations, we form the permutations of 3 elements by taking each of the 3 elements and associating with it the 2 permutations of the remaining two; we thus find that $P_3 = 3 \cdot 2 = 6$.

Similarly, to form the permutations of 4 elements we associate each of the 4 with the 6 permutations of the remaining 3: this gives $P_4 = 4 \cdot 3\,! = 4\,!$

This leads us to expect that $P_n = n\,!$ The actual proof rests on two facts: (a) the special fact, found by actual trial, that *e.g.* $P_2 = 2\,!$; (b) the general law that the number of permutations of $n+1$ elements is found by associating each of the $n+1$ elements with the P_n permutations of the remaining n, *i.e.* that

$$P_{n+1} = (n+1)P_n.$$

Knowing from (a) that $P_2 = 2\,!$ we find from this formula that $P_3 = 3 \cdot P_2 = 3 \cdot 2\,! = 3\,!$; in the same way that $P_4 = 4 \cdot 3\,! = 4\,!$ etc.

Notice that mathematical induction is not merely a method of trial and experiment. It requires that we should know not only one special case of the general formula to be proved, but also the law by which we can proceed from every special case to the next, *i.e.* from n to $n+1$ *whatever the value of n*. This law is a result, not of trial or induction, but of deductive reasoning. In our case it is expressed by the formula $P_{n+1} = (n+1)P_n$. The method of mathematical induction is therefore often called *reasoning from n to n* + 1.

63. Permutations by Groups. A somewhat more general problem in permutations is suggested by the following example: In an office there are two vacancies, one at \$ 1000, the other at \$ 800. There are 5 applicants for either of the 2 positions; in how many ways can the positions be filled?

The first vacancy can be filled in 5 ways, and then the second can still be filled in 4 ways; hence there are $5 \cdot 4 = 20$ ways. Denoting the applicants by a, b, c, d, e the 20 possibilities are:

$$\begin{array}{cccc} ab & ac & ad & ae \\ ba & bc & bd & be \\ ca & cb & cd & ce \\ da & db & dc & de \\ ea & eb & ec & ed \end{array}$$

The general problem here suggested is that of *finding the number of permutations of n elements k at a time*, where $k \leqq n$.

Each permutation here contains k elements; and we have to fill the k places in all possible ways from the n given elements. The first place can be filled in n ways. The second can then be filled in $n-1$ ways; hence the first and second places can be filled in $n(n-1)$ ways. The third place, when the first two are filled, can still be filled in $n-2$ ways, so that the first three places can be filled in $n(n-1)(n-2)$ ways. Proceeding in this way we find that the k places can be filled in $n\,(n-1)(n-2)\cdots(n-k+1)$ ways.

Thus the number of permutations of n elements, k at a time, which is denoted by ${}_nP_k$, is

$${}_nP_k = n(n-1)(n-2)\cdots(n-k+1).$$

Notice that in ${}_nP_k$ there are as many factors as places to be filled, viz. k: the first factor being n, the second $n-1$, etc., the kth will be $n-(k-1)=n-k+1$.

If $k=n$ we have the case of § 61: *i.e.* ${}_nP_n = P_n$.

As $n! = n(n-1)\cdots(n-k+1)\cdot(n-k)(n-k-1)\cdots 2\cdot 1 = n(n-1)\cdots(n-k+1)\cdot(n-k)!$, the expression for ${}_nP_k$ can also be written in the form

$${}_nP_k = \frac{n!}{(n-k)!}.$$

64. Combinations. If, in the problem of § 63, the 2 vacancies to be filled are positions of the same rank (as to salary, qualifications required, etc.), the answer will be different. We have now merely to select in all possible ways 2 out of 5 applicants, the arrangements *ab* and *ba*, *ac* and *ca*, etc., being now equivalent. Therefore the answer is now 20 divided by 2, *i.e.* 10, as can readily be verified directly: *ab, ac, ad, ae, bc, bd, be, cd, ce, de.*

If there were 3 vacancies, the number of ways of filling them from 4 applicants, when the positions are different, is ${}_4P_3 = 4 \cdot 3 \cdot 2 = 24$; but when the positions are alike, the number is 24 divided by the number of permutations of 3 things, *i.e.* $24/6 = 4$.

A set of k elements selected out of n, when the arrangement of the k elements in each set is indifferent, is called a ***combination***. The number of combinations of k elements that can be selected from n elements is denoted by ${}_nC_k$: to find this number we may first form the number ${}_nP_k$ of permutations of n elements k at a time, and then divide by the number $P_k = k!$ of permutations of k elements. Thus

$${}_nC_k = \frac{n(n-1)\cdots(n-k+1)}{1\cdot 2\cdots k} = \frac{n!}{k!\,(n-k)!}.$$

The number of combinations of n elements that can be selected from n elements is clearly 1; indeed, for $k = n$ our first expression gives ${}_nC_n = 1$.

EXERCISES

1. Find the value of n if

(*a*) $\dfrac{P_{n+1}}{P_n} = 5.$ (*b*) $\dfrac{P_{n-1}}{P_{n-3}} = 20.$ (*c*) $P_n = 40320.$

2. Show that

(*a*) ${}_nC_k = {}_nC_{n-k}$. (*b*) ${}_nC_k + {}_nC_{k-1} = {}_{n+1}C_k$. (*c*) $k\,{}_{n+1}C_k = (n+1)\,{}_nC_{k-1}$.

3. Prove by mathematical induction that:

(*a*) $1 + 2 + 3 + \cdots + n = \frac{1}{2}n(n+1)$.

(*b*) $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)$.

(*c*) $1^3 + 2^3 + 3^3 + \cdots + n^3 = [\frac{1}{2}n(n+1)]^2 = (1 + 2 + 3 + \cdots + n)^2$.

(*d*) $1 + 3 + 5 + \cdots + (2n-1) = n^2$.

(*e*) $2 + 4 + 6 + \cdots + 2n = n(n+1)$.

(*f*) $1\cdot 2 + 2\cdot 3 + 3\cdot 4 + \cdots + n(n+1) = \frac{1}{3}n(n+1)(n+2)$.

(*g*) $\dfrac{1}{1\cdot 2} + \dfrac{1}{2\cdot 3} + \dfrac{1}{3\cdot 4} + \cdots + \dfrac{1}{n(n+1)} = \dfrac{n}{n+1}.$

4. A pile of shot forms a pyramid with n shot on a side at the base. How many shot in the pile if the base is a square? an equilateral triangle?

5. Three football teams plan a series of games so that each team will play the other two teams 4 times. How many games in the schedule?

6. In how many ways can a committee of 3 freshmen and 2 sophomores be chosen from 8 freshmen and 5 sophomores?

7. In how many ways can the letters of the word *equal* be arranged in a row four letters at a time?

8. From the 26 letters of the alphabet, in how many ways can four different letters, one of which is d, be arranged in a row?

9. How many numbers of three digits each can be formed with 1, 2, 3, 4, 5, no digit being repeated? How many of these numbers are even? odd?

10. From a company of 60 men, how many guards of 4 men can be formed? How many times will one man (A) serve? How many times will A and B serve together?

11. Which is the largest of the numbers ${}_nC_1$, ${}_nC_2$, ${}_nC_3$, $\cdots$ ${}_nC_{n-1}$, when n is even? odd?

12. How many straight lines are determined by 12 points, no 3 of which are in a line?

13. How many triangles are determined by 10 points, no 3 of which are in a line?

65. Inversions in Permutations. When n elements a_1, a_2, a_3, $\cdots$ a_n, distinguished by their subscripts, are given, their arrangement, with the subscripts in the natural order of increasing numbers,

$$a_1a_2a_3 \cdots a_{n-1}a_n,$$

is called the *principal permutation*. In every other permutation of these elements it will occur that lower subscripts are preceded by higher ones. Every such occurrence is called an ***inversion***. *Any permutation is called **even** or **odd** according as the number of inversions occurring in it is even or odd.* The principal permutation, which has no inversion, is classed as even. To count the number of inversions in a given permutation, take each subscript in order and see by how many higher subscripts it is preceded. Thus, in the permutation

$$a_2a_3a_5a_1a_4a_6a_7,$$

the subscript 1 is preceded by the higher subscripts 2, 3, 5 (3 inversions); 2 and 3 are preceded by no higher subscripts; 4 is preceded by 5 (1 inversion); 5, 6, 7 are not preceded by any higher subscripts. Hence there are $3 + 1 = 4$ inversions, and the permutation is even. The permutation

$$a_6a_7a_3a_1a_2a_5a_4$$

of the same elements has $3 + 3 + 2 + 3 + 2 = 13$ inversions, and is, therefore, odd.

66. *If in a permutation any two adjacent elements are interchanged, the number of inversions is changed by* 1; hence the class to which the permutation belongs is changed (from even to odd or from odd to even).

Let the two adjacent elements be a_h, a_k and suppose that $h < k$. Two cases arise according as the original arrangement is a_ha_k or a_ka_h.

(*a*) If the original arrangement is a_ha_k, the new arrangement is a_ka_h; as $h < k$, and as all other elements of the permutation remain unchanged, the number of inversions is increased by 1.

(*b*) If the original arrangement is a_ka_h, the new arrangement is a_ha_k so that the number of inversions is diminished by 1.

67. *If in a permutation any two elements whatever be interchanged, the number of inversions is changed by an odd number*, and hence the class of the permutation is changed.

For, the interchange of any two elements a_h, a_k can be effected by a number of successive interchanges of adjacent elements. If there are m elements between a_h and a_k, we have only to interchange a_h with the first of these elements, then with the next, and so on, finally with a_k, and then a_k with the last of the m elements, with the next to the last, and so on; thus in all $m + 1 + m = 2m + 1$ interchanges of adjacent elements are required, *i.e.* an odd number.

68. *Of the $n!$ permutations of n elements just one half are even, the other half are odd.*

This follows by observing that if in each of the $n!$ permutations we interchange any two elements, the same in all, every even permutation becomes odd and every odd permutation becomes even, and no two different permutations are changed into the same permutation. After this interchange we must have exactly the same $n!$ permutations as before. Hence the number of even permutations must equal that of the odd permutations.

The propositions about inversions are important for the theory of determinants of the nth order to which we now proceed.

69. General Definition of Determinant. When n^2 numbers are given (*e.g.* the coefficients of the variables in n linear equations), arranged in a square array, we denote by the symbol

$$\begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \cdot & \cdot & \cdot \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}$$

and call *determinant of the* nth *order* the algebraic sum of the $n!$ terms obtained as follows: the first term is the product of the n numbers in the principal diagonal $a_{11}a_{22}a_{33} \cdots a_{nn}$; the other terms are derived from this term by permuting in all possible ways either the second subscripts or the first subscripts, and multiplying each term by $+1$ or -1 according as it is an even or odd permutation (*i.e.* contains an even or odd number of inversions).

It follows at once that *every term contains n factors, viz. one and only one from each row, and one and only one from each column.*

It is readily seen that this definition gives in the case of determinants of the second and third order the expressions previously used as defining such determinants. For a determinant of the fourth order,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}$$

we obtain the $1 \cdot 2 \cdot 3 \cdot 4 = 24$ terms from the *principal diagonal term*

$$a_{11}a_{22}a_{33}a_{44}$$

by forming all the permutations, say of the second subscripts 1, 2, 3, 4 and assigning the $+$ or $-$ sign according to the number of inversions. If these permutations are derived by successive interchanges of two subscripts the terms will have alternately the $+$ and $-$ sign.

70. The properties of the determinant of the nth order are essentially the same as those of the determinant of the third order (§§ 44–47).

(1) *The determinant is zero whenever all the elements of any row, or all those of any column, are zero.*

For, every term contains one element from each row and one from each column.

(2) It follows from the same observation that *if all elements of any row (or of any column) have a factor in common, this factor can be taken out and placed before the determinant.*

(3) *The value of a determinant is not changed by transposition; i.e.* by making the columns the rows, and *vice versa*, preserving their order.

For, this merely interchanges the subscripts of every element, *i.e.* the first series of subscripts becomes the second series, and *vice versa*.

Hence any property proved for rows is also true for columns.

(4) *The interchange of any two rows (columns) reverses the sign of the determinant.*

For, the interchange of any two rows gives an odd number of inversions to the first series of subscripts in the principal diagonal (§ 67), and does not alter the second series. Hence the signs of all the terms are reversed.

Cor. 1. *A determinant in which the elements of any row (column) are equal to the corresponding elements of any other row (column) is zero.*

For, the sign of the determinant is reversed when any two rows (columns) are interchanged; but the interchange of two *equal* rows (columns) cannot change the value of the determinant. Hence, denoting this value by A, we have in this case $-A = A$, *i.e.* $A = 0$.

(5) *If all the elements of any row (column) are sums of two terms, the determinant can be resolved into a sum of two determinants.*

For, in the expansion of the determinant every term contains one binomial factor; therefore it can be resolved into two terms. See § 47 for an illustration.

By means of this property, prove the following corollaries:

Cor. 1. *If all the elements of any row (column) are algebraic sums of any number of terms, the determinant can be resolved into a corresponding number of determinants.*

Cor. 2. *The value of a determinant is not changed by adding to the* *elements of any row (column) those of any other row (column) multiplied by any common factor.*

This corollary furnishes a method (see § 72) by which all the elements but one of any row (column) can be reduced to zero.

EXERCISES

1. How many inversions are there in the following permutations?

(*a*) $a_1a_4a_3a_5a_7a_2a_6$. (*b*) $a_7a_6a_1a_3a_2a_4a_5$. (*c*) $a_7a_6a_5a_4a_3a_2a_1$.

2. In the expansion of the determinant below, what sign must be placed before the terms *celn*, *agjp*?

$$\begin{vmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{vmatrix}.$$

3. Show that

$$\begin{vmatrix} a_1x + b_1y + c_1z & a_1 & b_1 & c_1 \\ a_2x + b_2y + c_2z & a_2 & b_2 & c_2 \\ a_3x + b_3y + c_3z & a_3 & b_3 & c_3 \\ a_4x + b_4y + c_4z & a_4 & b_4 & c_4 \end{vmatrix} = 0.$$

4. Reduce the following determinant to one in which all the elements of the first column are 1:

$$\begin{vmatrix} 2 & 4 & 1 & 3 \\ 3 & 7 & 5 & 6 \\ 2 & 0 & 0 & 5 \\ 6 & 1 & 2 & 3 \end{vmatrix}.$$

5. Show that

$$(a) \quad \begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} = 2\,abc(a+b+c)^3;$$

$$(b) \quad \begin{vmatrix} bb' + cc' & ba' & ca' \\ ab' & cc' + aa' & cb' \\ ac' & bc' & aa' + bb' \end{vmatrix} = 4\,aa'bb'cc'.$$

(Hint. Multiply the rows by *a*, *b*, *c*, respectively.)

71. Minors and Cofactors. If in a determinant both the row and column in which any particular element a_{ik} occurs be struck out, the remaining elements form a determinant of order $n-1$, which is called the *minor* of the element a_{ik}.

From the definition (§ 69), we observe that the expansion of any determinant is linear and homogeneous in the elements of any one row (column). The terms which contain a_{11} as factor are those terms whose other elements have all possible permutations of either the first or second subscripts 2, 3, ⋯ n. Hence the sum of the terms that contain a_{11} as factor can be expressed as a_{11} multiplied by its minor, *i.e.*

$$a_{11}\begin{vmatrix} a_{22} \cdots a_{2n} \\ \cdot \quad \cdot \quad \cdot \\ a_{n2} \cdots a_{nn} \end{vmatrix}.$$

By interchanging the first and second rows ((4) § 70) we observe similarly that the sum of those terms which contain a_{21} as factor can be written

$$-a_{21}\begin{vmatrix} a_{12} \cdots a_{1n} \\ \cdot \quad \cdot \quad \cdot \\ a_{n2} \cdots a_{nn} \end{vmatrix}.$$

Those terms which contain a_{31} as factor are given by

$$a_{31}\begin{vmatrix} a_{12} \cdots a_{1n} \\ \cdot \quad \cdot \quad \cdot \\ a_{n2} \cdots a_{nn} \end{vmatrix}$$

and so on. Hence the *expansion of a determinant by minors* of the first column is

$$a_{11}\begin{vmatrix} a_{22} \cdots a_{2n} \\ \cdot \quad \cdot \quad \cdot \\ a_{n2} \cdots a_{nn} \end{vmatrix} - a_{21}\begin{vmatrix} a_{12} \cdots a_{1n} \\ \cdot \quad \cdot \quad \cdot \\ a_{n2} \cdots a_{nn} \end{vmatrix} + \cdots (-1)^{n+1}a_{n1}\begin{vmatrix} a_{12} \quad \cdots \quad a_{1n} \\ \cdot \quad \cdot \quad \cdot \\ a_{n-1,\,2} \cdots a_{n-1,\,n} \end{vmatrix}.$$

Let A_{ik} denote the *cofactor* of a_{ik}; that is, $(-1)^{i+k}$ *times the minor of* a_{ik}; and A the original determinant; we can then write this expansion in the form

$$A = a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31} + \cdots + a_{n1}A_{n1}.$$

Similarly by cofactors of the elements of any column,

$$A = a_{1k}A_{1k} + a_{2k}A_{2k} + a_{3k}A_{3k} + \cdots + a_{nk}A_{nk}, \text{ for } k = 1, 2, 3, \cdots n,$$

and by cofactors of the elements of any row,

$$A = a_{i1}A_{i1} + a_{i2}A_{i2} + a_{i3}A_{i3} + \cdots + a_{in}A_{in}, \text{ for } i = 1, 2, 3, \cdots n.$$

The evaluation of a determinant of order n is thus reduced to the evaluation of n determinants of order $n-1$. To each of these the same process is applied until determinants of order 3 are obtained which can be evaluated by the rule of § 42.

72. In case of numerical determinants this process of successive reduction is very much simplified by reducing to zero all the elements of any one row (column) with the exception of one element, say a_{ik}. This can always be done by addition or subtraction of multiples of rows (columns), by Cor. 2, § 70. The expansion by cofactors of the elements of this row (column) then reduces to a single term, viz. $a_{ik}A_{ik}$.

The sign $(-1)^{i+k}$ to be affixed to the minor of a_{ik} to obtain the cofactor A_{ik} is readily found by counting plus, minus, plus, minus, etc., from the first element a_{11} down to the ith row and then to the kth column until a_{ik} is reached.

73. *The sum of the elements of any row (column) multiplied respectively by the cofactors of the elements of any other row (column) is zero.*

For, this corresponds to replacing the elements of any row (column) by the elements of another row (column). Hence the determinant vanishes (§ 70, (4), Cor. 1). For example, if in the expansion by cofactors of the first row

$$a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n}$$

we replace the elements of the first row by those of any other row we find

$$a_{i1}A_{11} + a_{i2}A_{12} + \cdots + a_{in}A_{1n} = 0, \quad \text{for } i = 2, 3, \cdots n.$$

74. Linear Equations. We write n equations in n variables $x_1, x_2, x_3, \cdots x_n$ as follows,

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= k_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= k_2, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad &\cdot \quad \cdot \quad \cdot \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= k_n. \end{aligned}$$

The determinant formed by the coefficients of the variables is called the determinant of the equations (§§ 37, 43) and is denoted by A. To solve the equations for any one of the variables, say x_j, we multiply the first equation by the cofactor of a_{1j} in A, *i.e.* by A_{1j}, the second equation by A_{2j}, the third equation by A_{3j}, etc., and add. This sum is by § 71

$$(a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj})x_j = Ax_j = k_1A_{1j} + k_2A_{2j} + \cdots + k_nA_{nj},$$

as the coefficients of all the other variables vanish (§ 73). Hence if

$A \neq 0$, we have the following rule: *Each variable is the quotient of two determinants, the denominator in each case is the determinant of the equations, while the numerator is obtained from the denominator by replacing the coefficients of the variable by the constant terms* (§§ 37, 43).

75. Elimination. If the n linear equations are homogeneous, *i.e.* if $k_1, k_2, \cdots k_n$ are all zero, we have

$$\begin{array}{c} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0, \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = 0. \end{array}$$

These equations are evidently satisfied by the values

$$x_1 = 0,\ x_2 = 0,\ \cdots x_n = 0.$$

Other values of the variables can satisfy the equations only if the determinant of the equations is zero. For, the method of § 74 gives in the case of homogeneous equations

$$Ax_1 = 0,\ Ax_2 = 0,\ \cdots Ax_n = 0.$$

Hence if $x_1, x_2, \cdots x_n$ are not *all* zero we must have

$$A = 0.$$

This result may also be stated as follows: *The result of eliminating n variables between n homogeneous linear equations is the determinant of the equations equated to zero.*

If, for instance, $x_n \neq 0$, we can divide each equation by x_n and then solve any $n - 1$ equations for the quotients $x_1/x_n, x_2/x_n, \cdots x_{n-1}/x_n$. It thus appears as in § 48 that when $A = 0$ the ratios of the variables can be found unless *all* the cofactors A_{ij} are zero.

EXERCISES

1. Show that

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ 0 & 0 & 1 & a_{34} \\ 0 & 0 & 0 & 1 \end{vmatrix}.$$

2. Write the expansion of

$$\begin{vmatrix} x & 0 & 0 & a_3 \\ -1 & x & 0 & a_2 \\ 0 & -1 & x & a_1 \\ 0 & 0 & -1 & a_0 \end{vmatrix}.$$

3. Express $a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4$ as a determinant.

4. Find the value of

$$\begin{vmatrix} a & b & c & d \\ -a & b & x & y \\ -a & -b & c & z \\ -a & -b & -c & d \end{vmatrix}.$$

5. Show that

$$\begin{vmatrix} 1+a & 1 & 1 & 1 & 1 \\ 1 & 1+b & 1 & 1 & 1 \\ 1 & 1 & 1+c & 1 & 1 \\ 1 & 1 & 1 & 1+d & 1 \\ 1 & 1 & 1 & 1 & 1+e \end{vmatrix} = abcde\left(1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e}\right).$$

6. Solve the equations:

$$(a)\ \begin{cases} 3x + y - z - 2w = -3, \\ 2x - y + 5z - 3w = 6, \\ 5x + 4y - z + w = 7, \\ x + 2y - 3z + w = -3. \end{cases} \qquad (b)\ \begin{cases} 4x - 2y + 2z + w = 1, \\ 2x + 3y - 3z + 3w = 2, \\ x - y + z - 4w = 4, \\ 3x + y - 4z + 3w = -5. \end{cases}$$

7. Are the following equations satisfied by other values of the variables than 0, 0, 0, 0?

$$(a)\ \begin{cases} 3x - 4y + 5z + w = 0, \\ 5x + 2y - 3z - w = 0, \\ x - y + z + w = 0, \\ 2x + 2y - 3z + 3w = 0. \end{cases} \qquad (b)\ \begin{cases} 3x + 2y + z - 5w = 0. \\ 9x + 9y + 6z - 10w = 0, \\ 2x + y - z + 3w = 0, \\ x + 2y + z + 4w = 0. \end{cases}$$

8. The relations between the sides and cosines of the angles of a triangle are $a = b\cos\gamma + c\cos\beta$, $b = c\cos\alpha + a\cos\gamma$, $c = a\cos\beta + b\cos\alpha$; find the relation between the cosines of the angles.

76. Special Forms.

In any determinant

$$\begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \cdot & \cdot & \cdot \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}$$

two elements are called *conjugate* when one occupies the same row and column that the other does column and row respectively; thus the element conjugate to a_{ik} is a_{ki}. The elements with equal subscripts a_{11}, a_{22}, $\cdots$ a_{nn} are called the *leading elements*; they are their own conjugates.

A determinant in which each element is equal to its conjugate (*i.e.* $a_{ik} = a_{ki}$) is called ***symmetric***.

A determinant in which each element is equal and opposite in sign to its conjugate (*i.e.* $a_{ik} = - a_{ki}$) is called ***skew-symmetric***; the condition implies that the leading elements are all zero.

A skew-symmetric determinant of odd order is always equal to zero.

For, if we change the rows to columns (§ 70, Prop. 3) and multiply each column by -1, the determinant resumes its original form. But since the determinant is of odd order we have multiplied by -1 an odd number of times, which changes the sign of the determinant [(4), § 70]. Hence denoting the value of the determinant by A, we have $-A = A$, *i.e.* $A = 0$.

77. Multiplication. It can easily be verified for determinants of the second order that the product of any two such determinants

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \cdot \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix}$$

can be expressed as a determinant of the second order in any one of the four following forms:

$$\begin{vmatrix} a_{11}b_{11} + a_{12}b_{12} & a_{11}b_{21} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{12} & a_{21}b_{21} + a_{22}b_{22} \end{vmatrix}; \quad \begin{vmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{vmatrix};$$

$$\begin{vmatrix} a_{11}b_{11} + a_{21}b_{21} & a_{11}b_{12} + a_{21}b_{22} \\ a_{12}b_{11} + a_{22}b_{21} & a_{12}b_{12} + a_{22}b_{22} \end{vmatrix}; \quad \begin{vmatrix} a_{11}b_{11} + a_{21}b_{12} & a_{11}b_{21} + a_{21}b_{22} \\ a_{12}b_{11} + a_{22}b_{12} & a_{12}b_{21} + a_{22}b_{22} \end{vmatrix}.$$

Thus the first of these forms is, by (5), § 70, equal to the sum of four determinants

$$\begin{vmatrix} a_{11}b_{11} & a_{11}b_{21} \\ a_{21}b_{11} & a_{21}b_{21} \end{vmatrix} + \begin{vmatrix} a_{11}b_{11} & a_{12}b_{22} \\ a_{21}b_{11} & a_{22}b_{22} \end{vmatrix} + \begin{vmatrix} a_{12}b_{12} & a_{11}b_{21} \\ a_{22}b_{12} & a_{21}b_{21} \end{vmatrix} + \begin{vmatrix} a_{12}b_{12} & a_{12}b_{22} \\ a_{22}b_{12} & a_{22}b_{22} \end{vmatrix}$$

of which the first and fourth are zero, while the sum of the second and third reduces to

$$b_{11}b_{22}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} - b_{12}b_{21}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \cdot \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix}.$$

For determinants of higher order the same method can be shown to hold. Without giving the general proof we here confine ourselves to illustrating the method for determinants of the third order:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \cdot \begin{vmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{vmatrix} = \begin{vmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{vmatrix},$$

where

$$c_{11} = a_{11}b_{11} + a_{12}b_{12} + a_{13}b_{13}, \quad c_{12} = a_{11}b_{21} + a_{12}b_{22} + a_{13}b_{23},$$
$$c_{13} = a_{11}b_{31} + a_{12}b_{32} + a_{13}b_{33}, \quad c_{21} = a_{21}b_{11} + a_{22}b_{12} + a_{23}b_{13},$$

etc. The product determinant can here also be written in four different forms, according as we combine rows with rows, rows with columns, columns with rows, or columns with columns.

If the two determinants to be multiplied are not of the same order, they can be made of the same order by adding to the lower determinant columns and rows consisting of zeros and a one; thus

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{vmatrix}, \text{ etc.}$$

EXERCISES

1. Show that (*a*) The minors of the leading elements of a symmetric determinant are symmetric. (*b*) The minors of the leading elements of a skew-symmetric determinant are skew-symmetric. (*c*) The square of any determinant is a symmetric determinant.

2. Expand the symmetric determinants:

$$(a)\ \begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix}. \qquad (b)\ \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & x & y \\ 1 & x & 0 & z \\ 1 & y & z & 0 \end{vmatrix}. \qquad (c)\ \begin{vmatrix} 1 & x & y & z \\ x & 1 & 0 & 0 \\ y & 0 & 1 & 0 \\ z & 0 & 0 & 1 \end{vmatrix}.$$

$$(d)\ \begin{vmatrix} 1 & p & x+p \\ p & q & px+q \\ x+p & px+q & 0 \end{vmatrix}. \qquad (e)\ \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix}.$$

3. Show that

$$(a)\ \begin{vmatrix} 0 & 1 & 1 \\ 1 & a & b \\ 1 & c & d \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 \\ 1 & a+\alpha & b+\beta \\ 1 & c+\alpha & d+\beta \end{vmatrix}. \quad \text{(State this property in words.)}$$

(b) $\begin{vmatrix} x & a & a \\ a & x & a \\ a & a & x \end{vmatrix} = (x-a)^2(x+2a).$ (c) $\begin{vmatrix} x & a & a & a \\ a & x & a & a \\ a & a & x & a \\ a & a & a & x \end{vmatrix} = (x-a)^3(x+3a).$

4. Show that any determinant whose elements on either side of the principal diagonal are all zero, is equal to the product of the leading elements.

5. A symmetric determinant in which all the elements of the first row and first column are 1 and such that every other element is the sum of the element above and the element to the left of it, has the value 1. Illustrate this proposition for a determinant of the fourth order.

6. Show that any skew-symmetric determinant of order 2 or 4 is a perfect square. This is true for any skew-symmetric determinant of even order.

7. Expand the following determinants:

(a) $\begin{vmatrix} 1 & a & -b \\ -a & 1 & c \\ b & -c & 1 \end{vmatrix}.$ (b) $\begin{vmatrix} 0 & a & b & c \\ -a & 0 & f & e \\ -b & -f & 0 & d \\ -c & -e & -d & 0 \end{vmatrix}.$ (c) $\begin{vmatrix} 0 & a & -b & c \\ -a & 0 & f & e \\ b & -f & 0 & d \\ -c & -e & -d & 0 \end{vmatrix}.$

8. Express as a determinant

(a) $\begin{vmatrix} a & b \\ c & d \end{vmatrix}^2.$ (b) $\begin{vmatrix} 0 & x & y \\ x & 0 & z \\ y & z & 0 \end{vmatrix}^2.$ (c) $\begin{vmatrix} 3 & 2 & 1 \\ 4 & 2 & 3 \\ 7 & 1 & 5 \end{vmatrix} \cdot \begin{vmatrix} 2 & -1 \\ 1 & 3 \end{vmatrix}.$

(d) $\begin{vmatrix} x & a & a \\ a & x & a \\ a & a & x \end{vmatrix} \cdot \begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix}.$ (e) $\begin{vmatrix} x & 0 & z \\ x & y & 0 \\ 0 & y & z \end{vmatrix} \cdot \begin{vmatrix} u & 0 & w \\ u & v & 0 \\ 0 & v & w \end{vmatrix}.$

(f) $\begin{vmatrix} a_{11}-s & a_{12} & a_{13} \\ a_{21} & a_{22}-s & a_{23} \\ a_{31} & a_{32} & a_{33}-s \end{vmatrix} \cdot \begin{vmatrix} a_{11}+s & a_{12} & a_{13} \\ a_{21} & a_{22}+s & a_{23} \\ a_{31} & a_{32} & a_{33}+s \end{vmatrix}.$

9. Show that

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix} \cdot \begin{vmatrix} a' & b' & c' \\ d' & e' & f' \\ g' & h' & k' \end{vmatrix} = \begin{vmatrix} a & b & c & 0 & 0 & 0 \\ d & e & f & 0 & 0 & 0 \\ g & h & k & 0 & 0 & 0 \\ \alpha & \beta & \gamma & a' & b' & c' \\ \delta & \epsilon & \zeta & d' & e' & f' \\ \eta & \theta & \kappa & g' & h' & k' \end{vmatrix}.$$

CHAPTER VI

THE CIRCLE. QUADRATIC EQUATIONS

78. Circles. A circle, in a given plane, is defined as *the locus of all those points of the plane which are at the same distance from a fixed point.*

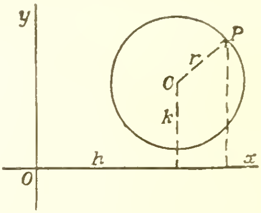


Fig. 31

Let C (h, k) be the center, r the radius (Fig. 31); the necessary and sufficient condition that any point P (x, y) is at the distance r from C (h, k) is that

$$(1)\qquad (x-h)^2+(y-k)^2=r^2.$$

This equation, which is satisfied by the coordinates x, y of every point on the circle, and by the coordinates of no other point, is called *the equation of the circle of center C (h, k) and radius r.*

If the center of the circle is at the origin O $(0, 0)$, the equation of the circle is evidently

$$(2)\qquad x^2+y^2=r^2.$$

EXERCISES

Write down the equations of the following circles:

(*a*) center (3, 2), radius 7;

(*b*) center at origin, radius 3;

(*c*) center at $(-a, 0)$, radius a;

(*d*) circle of any radius touching the axis Ox at the origin;

(*e*) circle of any radius touching the axis Oy at the origin.

Illustrate each case by a sketch.

79. Equation of Second Degree. Expanding the equation (1) of § 78, we obtain the equation of the circle in the new form

$$x^2 + y^2 - 2hx - 2ky + h^2 + k^2 - r^2 = 0.$$

This is an *equation of the second degree* in x and y. But it is of a particular form. ***The general equation of the second degree*** in x and y is of the form

$$Ax^2 + 2Hxy + By^2 + 2Gx + 2Fy + C = 0; \tag{3}$$

i.e. it contains a constant term, C; two terms of the first degree, one in x and one in y; and three terms of the second degree, one in x^2, one in xy, and one in y^2.

If in this general equation we have

$$H = 0, \quad B = A \neq 0,$$

it reduces, upon division by A, to the form

$$x^2 + y^2 + \frac{2G}{A}x + \frac{2F}{A}y + \frac{C}{A} = 0,$$

which agrees with the form (1) of the equation of a circle, except for the notation for the coefficients.

We can therefore say that *any equation of the second degree which contains no xy-term and in which the coefficients of x^2 and y^2 are equal, may represent a circle.*

80. Determination of Center and Radius. To draw the circle represented by the general equation

$$\boldsymbol{Ax^2 + Ay^2 + 2Gx + 2Fy + C = 0,} \tag{4}$$

where A, G, F, C are any real numbers while $A \neq 0$, we first divide by A and *complete the squares in x and y; i.e.* we first write the equation in the form

$$\left(x + \frac{G}{A}\right)^2 + \left(y + \frac{F}{A}\right)^2 = \frac{G^2}{A^2} + \frac{F^2}{A^2} - \frac{C}{A}.$$

The left-hand member represents the square of the distance of the point (x, y) from the point $(-G/A, -F/A)$; the right-hand member is constant. The given equation therefore represents the circle whose center has the coordinates

$$h = -\frac{G}{A},\ k = -\frac{F}{A},$$

and whose radius is

$$r = \sqrt{\frac{G^2}{A^2} + \frac{F^2}{A^2} - \frac{C}{A}} = \frac{1}{A}\sqrt{G^2 + F^2 - AC}.$$

This radius is, however, imaginary if $G^2 + F^2 < AC$; in this case the equation is not satisfied by any points with real coordinates.

If $G^2 + F^2 = AC$, the radius is zero, and the equation is satisfied only by the coordinates of the point $(-G/A, -F/A)$.

If $G^2+F^2 > AC$, the radius is real, and the equation represents a real circle.

Thus, *the general equation of the second degree* (3), § 79, *represents a circle if, and only if,*

$$A = B \neq 0,\ H = 0,\ G^2 + F^2 > AC.$$

81. Circle determined by Three Conditions. The equation (1) of the circle contains three constants h, k, r. The general equation (4) contains four constants of which, however, only three are essential since we can always divide through by one of these constants. Thus, dividing by A and putting $2\,G/A = a$, $2\,F/A = b$, $C/A = c$, the general equation (4) assumes the form

$$(5) \qquad x^2 + y^2 + ax + by + c = 0,$$

with the three constants a, b, c.

The existence of three constants in the equation corresponds to the possibility of determining a circle geometrically, in a variety of ways, by three conditions. It should be remembered in this connection that the equation of a straight line contains two essential constants, the line being determined by two geometrical conditions (§ 30).

EXERCISES

1. Draw the circles represented by the following equations:

(*a*) $2x^2 + 2y^2 - 8x + 5y + 1 = 0$. (*b*) $3x^2 + 3y^2 + 17x - 15y - 6 = 0$.
(*c*) $4x^2 + 4y^2 - 6x - 10y + 4 = 0$. (*d*) $x^2 + y^2 + x - 4y = 0$.
(*e*) $2x^2 + 2y^2 - 7x = 0$. (*f*) $x^2 + y^2 - 3x - 6 = 0$.

2. What is the equation of the circle of center (h, k) that touches the axis Ox? that touches the axis Oy? that passes through the origin?

3. What is the equation of any circle whose center lies on the axis Ox? on the axis Oy? on the line $y = x$? on the line $y = 2x$? on the line $y = mx$?

4. Find the equation of the circle whose center is at the point $(-4, 6)$ and which passes through the point $(2, 0)$.

5. Find the circle that has the points $(4, -3)$ and $(-2, -1)$ as ends of a diameter.

6. A swing moving in the vertical plane of the observer is 48 ft. away and is suspended from a pole 27 ft. high. If the seat when at rest is 2 ft. above the ground, what is the equation of the path (for the observer as origin)? What is the distance of the seat from the observer when the rope is inclined at 45° to the vertical?

7. Find the locus of a point whose distance from the point (a, b) is κ times its distance from the origin.

Let $P(x, y)$ be *any* point of the locus; then the condition is

$$\sqrt{(x-a)^2 + (y-b)^2} = \kappa\sqrt{x^2 + y^2};$$

upon squaring and rearranging this becomes:

$$(1-\kappa^2)x^2 + (1-\kappa^2)y^2 - 2ax - 2by + a^2 + b^2 = 0.$$

Hence for any value of κ except $\kappa = 1$, the locus is a circle whose center is $a/(1-\kappa^2)$, $b/(1-\kappa^2)$ and whose radius is $\kappa\sqrt{a^2+b^2}/(1-\kappa^2)$. What is the locus when $\kappa = 1$?

8. Find the locus of a point twice as far from the origin as from the point $(6, -3)$. Sketch.

9. What is the locus of a point whose distances from two points P_1, P_2 are in the constant ratio κ?

10. Determine the locus of the points which are κ times as far from the point $(-2, 0)$ as from the point $(2, 0)$. Assign to κ the values $\sqrt{5}$, $\sqrt{3}$, $\sqrt{2}$, $\frac{1}{5}\sqrt{5}$, $\frac{1}{3}\sqrt{3}$, $\frac{1}{2}\sqrt{2}$ and illustrate with sketches drawn with respect to the same axes.

11. Determine the locus of a point whose distance from the line $3x - 4y + 1 = 0$ is equal to the square of its distance from the origin. Illustrate with a sketch.

12. Determine the locus of a point if the square of its distance from the line $x + y - a = 0$ is equal to the product of its distances from the axes.

82. Circle in Polar Coordinates. Let us now express the *equation of a circle in polar coordinates.* If $C(r_1, \phi_1)$ is the center of a circle of radius a (Fig. 32) and $P(r, \phi)$ any point of the circle, then by the cosine law of trigonometry

$$r^2 + r_1^2 - 2\, r_1 r \cos(\phi - \phi_1) = a^2.$$

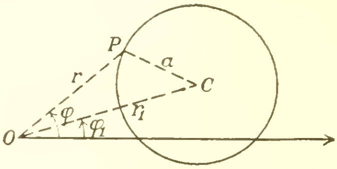


FIG. 32

This is the equation of the circle since, for given values of r_1, ϕ_1, a, it is satisfied by the coordinates r, ϕ of every point of the circle, and by the coordinates of no other point.

Two special cases are important:

(1) If the origin O be taken on the circumference and the polar axis along a diameter OA (Fig. 33), the equation becomes

$$r^2 + a^2 - 2\, ar \cos\phi = a^2,$$

i.e.
$$r = 2\, a \cos\phi.$$

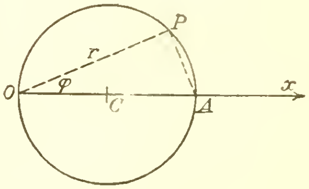


FIG. 33

This equation has a simple geometrical interpretation: the radius vector of any point P on the circle is the projection of the diameter $OA = 2\,a$ on the direction of the radius vector.

(2) If the origin be taken at the center of the circle, the equation is
$$r = a.$$

EXERCISES

1. Draw the following circles in polar coordinates:

(*a*) $r = 10 \cos \phi$. (*b*) $r = 2a \cos (\phi - \frac{1}{3}\pi)$. (*c*) $r = \sin \phi$.
(*d*) $r = 6$. (*e*) $r = 7 \sin (\phi - \frac{1}{6}\pi)$. (*f*) $r = 17 \cos \phi$.

2. Write the equation of the circle in polar coordinates:

(*a*) with center at $(10, \frac{1}{2}\pi)$ and radius 5;
(*b*) with center at $(6, \frac{1}{4}\pi)$ and touching the polar axis;
(*c*) with center at $(4, \frac{3}{2}\pi)$ and passing through the origin;
(*d*) with center at $(3, \pi)$ and passing through the point $(4, \frac{1}{3}\pi)$.

3. Change the equations of Ex. (1) and (2) to rectangular coordinates with the origin at the pole and the axis Ox coincident with the polar axis.

4. Determine in polar coordinates the locus of the midpoints of the chords drawn from a fixed point of a circle.

83. Quadratic Equations. The fundamental problem of finding the intersections of a line and a circle leads, as we shall see (§ 86), to a quadratic equation. Before discussing it we here recall briefly the essential facts about quadratic equations.

The method for solving a quadratic equation consists in completing the square of the terms in x^2 and x, which is done most conveniently after dividing the equation by the coefficient of x^2. The equation

$$x^2 + 2px + q = 0$$

has the roots

$$x = -p \pm \sqrt{p^2 - q}.$$

The quantity $p^2 - q$ is called the ***discriminant*** of the equation. According as the discriminant is positive, zero, or negative, the roots are real and different, real and equal, or imaginary. In the last case, *i.e.* when $p^2 < q$, the roots are, more precisely, *conjugate complex*, *i.e.* of the form $a + bi$ and $a - bi$, where a and b are real while $i = \sqrt{-1}$.

As remarked above, any quadratic equation may be thrown into the form here discussed, by dividing by the coefficient of x^2.

84. Relations between Roots and Coefficients. If we denote the roots of the quadratic equation

$$x^2 + 2px + q = 0$$

by x_1 and x_2, we have

$$x_1 = -p + \sqrt{p^2 - q}, \qquad x_2 = -p - \sqrt{p^2 - q},$$

whence

$$x_1 + x_2 = -2p, \quad x_1x_2 = q;$$

i.e. *the sum of the roots of a quadratic equation in which the coefficient of x^2 is reduced to 1 is equal to minus the coefficient of x; the product of the roots is equal to the constant term.*

With the values of x_1, x_2 just given we find

$$(x - x_1)(x - x_2) = x^2 + 2px + q,$$

so that the quadratic equation can be written in the form

$$(x - x_1)(x - x_2) = 0,$$

which gives

$$x^2 - (x_1 + x_2)x + x_1x_2 = 0.$$

These properties of the roots often make it possible to solve a quadratic equation by inspection.

EXERCISES

1. Solve the quadratic equations:

(*a*) $x^2 - 6x + 8 = 0$.
(*b*) $x^2 + 5x - 14 = 0$.
(*c*) $2x^2 - x - 28 = 0$.
(*d*) $5x^2 - 7x - 6 = 0$.
(*e*) $x^2 + 2bx - a^2 + b^2 = 0$.
(*f*) $a^2x^2 - (a^2 + b^2)x + b^2 = 0$.
(*g*) $ax^2 + bx = 0$.
(*h*) $12x^2 + 8x - 15 = 0$.

2. Show that the solutions of the quadratic equation $ax^2 + bx + c = 0$ may be written in the form $x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$.

When are these solutions real and unequal? equal? imaginary?

3. Write down the quadratic equation that has the following roots:

(*a*) $3, -2$.
(*b*) $-3, 0$.
(*c*) $5, -5$.
(*d*) $a - b, a + b$.
(*e*) $3 - 2\sqrt{3}, 3 + 2\sqrt{3}$.
(*f*) $1 + \sqrt{2}, 1 - \sqrt{2}$.
(*g*) $c, -\frac{1}{c}$.
(*h*) $\frac{1}{2}, -\frac{1}{3}$.
(*i*) $3, \sqrt{2}$.

4. Without solving, determine the nature of the roots of the following equations :

(*a*) $5x^2-6x-2=0.$ (*b*) $9x^2+6x+1=0.$

(*c*) $2x^2-x+3=0.$ (*d*) $20x^2+6x-5=0.$

(*e*) $11x^2-4x-\frac{1}{22}=0.$ (*f*) $3x^2+2x+1=0.$

5. For what values of k are the roots of the following equations real and different ? real and equal ? conjugate complex ?

(*a*) $x^2-4x+k=0.$ (*b*) $x^2+2kx+36=0.$

(*c*) $9x^2+kx+25=0.$ (*d*) $ax^2+bx+k=0.$

(*e*) $kx^2-5x+6=0.$ (*f*) $ax^2+kx+c=0.$

6. Solve the following equations as quadratic equations :

(*a*) $y^4-3y^2-4=0.$ (Let $y^2=z.$) (*b*) $2z^{-2}+3z^{-1}-2=0.$

(*c*) $x+\sqrt{x+3}=3.$ (*d*) $\dfrac{2}{x+3}+\dfrac{x+3}{2}=2.$

(*e*) $m^6+18m^3-243=0.$ (*f*) $2x^{-\frac{2}{3}}+x^{-\frac{1}{3}}-15=0.$

7. If x_1 and x_2 are the roots of $x^2+2px+q=0$, find the values of

(*a*) $x_1^2x_2+x_1x_2^2.$ (*b*) $x_1^2+x_2^2.$ (*c*) $(x_1-x_2)^2.$

(*d*) $\dfrac{1}{x_1}+\dfrac{1}{x_2}.$ (*e*) $\dfrac{1}{x_1^2}+\dfrac{1}{x_2^2}$ (*f*) $x_1^3+x_2^3.$

and apply these results to the case $x^2-3x+4=0.$

8. Without solving, form the equation whose roots are each twice the roots of $x^2-3x+7=0.$ [See § 84.]

9. What is the equation whose roots are m times the roots of $x^2+2px+q=0$?

10. Form the equation whose roots are related to the roots of $2x^2-3x-5=0$, in the following ways :

(*a*) less by 2 ; (*b*) greater by 3 ; (*c*) divided by 6.

85. Simultaneous Linear and Quadratic Equations. To solve two equations in x and y of which one is of the first degree (linear) while the other is of the second degree, it is generally most convenient to *solve the linear equation for either x or y and to substitute the value so found in the equation of the second degree. It then remains to solve a quadratic equation.*

An equation of the first degree represents a straight line.

If the given equation of the second degree be of the form described in § 79, it will represent a circle. By solving two such simultaneous equations we find the coordinates of the points that lie both on the line and on the circle, *i.e. the points of intersection of line and circle.*

86. Intersection of Line and Circle. Let us find the intersections of the line

$$y = mx + b$$

with the circle about the origin

$$x^2 + y^2 = r^2.$$

Substituting the value of y from the former equation into the latter, we find the quadratic equation in x:

$$x^2 + (mx + b)^2 = r^2,$$

or

$$(1 + m^2)x^2 + 2\,mbx + b^2 - r^2 = 0.$$

The two roots x_1, x_2 of this equation are the abscissas of the points of intersection; the corresponding ordinates are found by substituting x_1, x_2 in $y = mx + b$.

It is easily seen that the abscissas x_1, x_2 are real and different if

$$(1 + m^2)r^2 - b^2 > 0,$$

i.e. if

$$\frac{b}{\sqrt{1 + m^2}} < r.$$

Since $m = \tan\alpha$, and hence $1/\sqrt{1 + m^2} = \cos\alpha$, the preceding relation means that $b\cos\alpha < r$, *i.e.* the line has a distance from the origin less than the radius of the circle. If

$$(1 + m^2)r^2 - b^2 = 0,$$

the roots x_1, x_2 are real and equal. The line and the circle then have only a single point in common. Such a line is said to *touch* the circle or to be a ***tangent*** to the circle. If

$$(1 + m^2)r^2 - b^2 < 0,$$

the roots are complex, and the line has no points in common with the circle.

87. The General Case. The intersections of the line and circle

$$Ax + By + C = 0,$$
$$x^2 + y^2 + ax + by + c = 0,$$

are found in the same way: substitute the value of y (or x), found from the equation of the line, in the equation of the circle and solve the resulting quadratic equation.

It is often desired to determine merely *whether the line is tangent to the circle.* To answer this question, substitute y (or x) from the linear equation in the equation of the circle and, *without solving the quadratic equation*, write down the condition for equal roots ($p^2 = q$, § 83).

EXERCISES

1. Find the coordinates of the points where the circle $x^2 + y^2 - x + y - 12 = 0$ crosses the axes.

2. Find the intersections of the line $3x + y - 5 = 0$ and the circle $x^2 + y^2 - 22x - 4y + 25 = 0$.

3. Find the intersections of the line $2x - 7y + 5 = 0$ and the circle $2x^2 + 2y^2 + 9x + 9y - 11 = 0$.

4. Find the equations of the tangents to the circle $x^2 + y^2 = 16$ that are parallel to the line $y = -3x + 8$.

5. Show that the equations of the tangents to the circle $x^2 + y^2 = r^2$ with slope m are $y = mx \pm r\sqrt{1 + m^2}$.

6. For what value of r will the line $3x - 2y - 5 = 0$ be tangent to the circle $x^2 + y^2 = r^2$?

7. Find the equations of the tangents to the circle $2x^2 + 2y^2 - 3x + 5y - 7 = 0$ that are perpendicular to the line $x + 2y + 3 = 0$.

8. Find the midpoint of the chord intercepted by the line $5x - y + 9 = 0$ on the circle $x^2 + y^2 = 18$. (Use § 84.)

9. Find the equations of the tangents to the circle $x^2 + y^2 = 58$ that pass through the point (10, 4).

88. The Tangent to a Circle. The tangent to a circle (compare § 86) at any point P may be defined as the perpendicular through P to the radius passing through P. To find the equation of the tangent to a circle whose center is at the origin,

$$x^2 + y^2 = r^2,$$

at the point $P(x, y)$ of the circle (Fig. 34), observe that the distance p of the tangent from the origin is equal to the radius r and that the angle β made by this distance with the axis Ox is such that

$$\cos\beta = \frac{x}{r}, \quad \sin\beta = \frac{y}{r};$$

substituting these values in the normal form $X\cos\beta + Y\sin\beta = p$ of the equation of a line (§ 54), we find as equation of the tangent

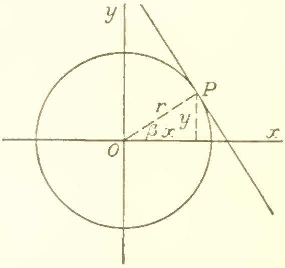


FIG. 34

$$xX + yY = r^2,$$

where x, y are the coordinates of the point of contact P and X, Y are those of any point of the tangent.

89. The General Case. To find the equation of the tangent to a circle whose center is not at the origin let us write the general equation (4), § 80, *viz.*

$$(4) \qquad Ax^2 + Ay^2 + 2Gx + 2Fy + C = 0,$$

in the form

$$\left(x + \frac{G}{A}\right)^2 + \left(y + \frac{F}{A}\right)^2 = \frac{G^2}{A^2} + \frac{F^2}{A^2} - \frac{C}{A},$$

where $-G/A$, $-F/A$ are the coordinates of the center and $G^2/A^2 + F^2/A^2 - C/A$ is the square of the radius r (§ 80). With respect to parallel axes through the center the same circle has the equation

$$x^2 + y^2 = \frac{G^2}{A^2} + \frac{F^2}{A^2} - \frac{C}{A} = r^2.$$

and the tangent at the point $P(x, y)$ of the circle is (§ 88):

$$xX + yY = \frac{G^2}{A^2} + \frac{F^2}{A^2} - \frac{C}{A} = r^2.$$

Hence, transferring back to the original axes, we find as equation of the tangent at $P(x, y)$ to the circle (4):

$$AxX + AyY + G(x + X) + F(y + Y) + C = 0.$$

This general form of the tangent is readily remembered if we observe that it can be derived from the equation (4) of the circle by replacing x^2 by xX, y^2 by yY, $2x$ by $x + X$, $2y$ by $y + Y$.

EXERCISES

1. Find the tangent to the given circle at the given point:

(*a*) $x^2 + y^2 = 41$, $(5, -4)$.
(*b*) $x^2 + y^2 + 6x + 5y - 16 = 0$, $(-2, 3)$.
(*c*) $3x^2 + 3y^2 + 10x + 17y + 18 = 0$, $(-2, -5)$.
(*d*) $x^2 + y^2 - ax - by = 0$, (a, b).

2. The equation of any circle through the origin can be written in the form (§ 81) $x^2 + y^2 + ax + by = 0$; show that the line $ax + by = 0$ is the tangent at the origin, and find the equation of the parallel tangent.

3. Derive the equation of the tangent to the circle $(x-h)^2 + (y-k)^2 = r^2$.

4. Show that the circles $x^2 + y^2 - 6x + 2y + 2 = 0$ and $x^2 + y^2 - 4y + 2 = 0$ touch at the point $(1, 1)$.

5. Find the tangents to the circle $x^2 + y^2 - 2x - 10y + 9 = 0$ at the extremities of the diameter through the point $(-1, 11/2)$.

6. The line $2x + y = 10$ is tangent to the circle $x^2 + y^2 = 20$; what is the point of contact?

7. What is the point of contact if $Ax + By + C = 0$ is tangent to the circle $x^2 + y^2 = r^2$?

8. Show that $x - y - 1 = 0$ is tangent to the circle $x^2 + y^2 + 4x - 10y - 3 = 0$, and find the point of contact.

9. By § 86, the line $y = mx + b$ has but one point in common with the circle $x^2 + y^2 = r^2$ if $(1 + m^2)r^2 = b^2$; show that in this case the radius drawn to the common point is perpendicular to the line $y = mx + b$.

90. Circle through Three Points. *To find the equation of the circle passing through three points* $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, $P_3(x_3, y_3)$, observe that the coordinates of these points satisfy the equation of the circle (§ 81)

$$x^2 + y^2 + ax + by + c = 0; \tag{6}$$

hence we must have

$$\begin{cases} x_1^2 + y_1^2 + ax_1 + by_1 + c = 0, \\ x_2^2 + y_2^2 + ax_2 + by_2 + c = 0, \\ x_3^2 + y_3^2 + ax_3 + by_3 + c = 0. \end{cases} \tag{7}$$

From the last three equations we can find the values of a, b, and c; these values must then be substituted in the first equation.

In general this is a long and tedious operation. What we actually wish to do is to eliminate a, b, c between the *four* equations above. The theory of determinants furnishes a very simple means of eliminating four quantities between four *homogeneous* linear equations (§ 75). Our equations are *not* homogeneous in a, b, c. But if we write the first two terms in each equation with the factor 1: $(x^2 + y^2) \cdot 1$, $(x_1^2 + y_1^2) \cdot 1$, etc., we have four equations which are linear and homogeneous in 1, a, b, c; hence the result of eliminating these four quantities is the determinant of their coefficients equated to zero. Thus the *equation of the circle through three points* is

$$\begin{vmatrix} x^2 + y^2 & x & y & 1 \\ x_1^2 + y_1^2 & x_1 & y_1 & 1 \\ x_2^2 + y_2^2 & x_2 & y_2 & 1 \\ x_3^2 + y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0.$$

Compare § 49, where the equation of the straight line through two points is given in determinant form.

EXERCISES

1. Find the equations of the circles that pass through the points:

(*a*) $(2, 3)$, $(-1, 2)$, $(0, -3)$.
(*b*) $(0, 0)$, $(1, -4)$, $(5, 0)$.
(*c*) $(0, 0)$, $(a, 0)$, $(0, b)$.

2. Find the circles through the points $(3, -1)$, $(-1, -2)$ which touch the axis Ox.

3. Find the circle through the points $(2, 1)$, $(-1, 3)$ with center on the line $3x - y + 2 = 0$.

4. Find the circle whose center is $(3, -2)$ and which touches the line $3x + 4y - 12 = 0$.

5. Find the circle through the origin that touches the line

$$4x - 5y - 14 = 0 \text{ at } (6, 2).$$

6. Find the circle inscribed in the triangle determined by the lines

$$24x - 7y + 3 = 0,\ 3x - 4y - 9 = 0,\ 5x + 12y - 50 = 0.$$

7. Two circles are said to be *orthogonal* if their tangents at a point of intersection are perpendicular; the square of the distance between their centers is then equal to the sum of the squares of their radii. If the equations of two intersecting circles are

$$x^2 + y^2 + a_1x + b_1y + c_1 = 0, \text{ and } x^2 + y^2 + a_2x + b_2y + c_2 = 0,$$

show that the circles are orthogonal when $a_1a_2 + b_1b_2 = 2(c_1 + c_2)$.

8. Find the circle that has its center at $(-2, 1)$ and is orthogonal to the circle $x^2 + y^2 - 6x + 3 = 0$.

9. Find the circle that has its center on the line $y = 3x + 4$, passes through the point $(4, -3)$, and is orthogonal to the circle

$$x^2 + y^2 + 13x + 5y + 2 = 0.$$

91. Inversion. A circle of center O and radius a being given (Fig. 35), we can find to every point P of the plane (excepting the center O) one and only one point P' on OP, produced beyond P if necessary, such that

$$OP \cdot OP' = a^2.$$

The point P' is said to be ***inverse*** to P with respect to the circle (O, a); and as the relation is not

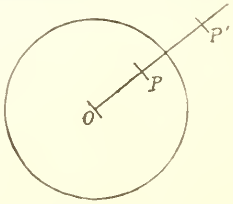


FIG. 35

changed by interchanging P and P', the point P is inverse to P'. The point O is called the *center of inversion.*

It is clear that (1) the inverse of a point P within the circle is a point P' without, and *vice versa;* (2) the inverse of a point of the circle itself coincides with it; (3) as P approaches the center O, its inverse P' moves off to infinity, and *vice versa.*

The *inverse of any geometrical figure* (line, curve, area, etc.) is the figure formed by the points inverse to all the points of the given figure.

92. Inverse of a Circle.

Taking rectangular axes through O (Fig. 36), we find for the relations between the coordinates of two inverse points $P(x, y)$, $P'(x', y')$, if we put $OP = r$, $OP' = r'$:

$$\frac{x'}{x} = \frac{y'}{y} = \frac{r'}{r} = \frac{rr'}{r^2} = \frac{a^2}{r^2}$$

since $rr' = a^2$; hence

$$x' = \frac{a^2x}{x^2 + y^2}, \quad y' = \frac{a^2y}{x^2 + y^2};$$

and similarly

$$x = \frac{a^2x'}{x'^2 + y'^2}, \quad y = \frac{a^2y'}{x'^2 + y'^2}.$$

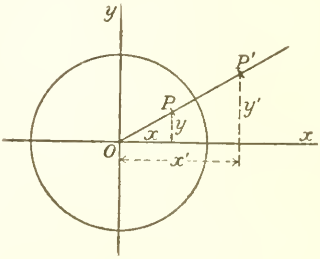


FIG. 36

These equations enable us to find to any curve whose equation is given the equation of the inverse curve, by simply substituting for x, y their values.

Thus it can be shown that *by inversion any circle is transformed into a circle or a straight line.*

For, if in the general equation of the circle

$$A(x^2 + y^2) + 2\,Gx + 2\,Fy + C = 0$$

we substitute for x and y the above values, we find

$$Aa^4\frac{x'^2 + y'^2}{(x'^2 + y'^2)^2} + 2\,Ga^2\frac{x'}{x'^2 + y'^2} + 2\,Fa^2\frac{y'}{x'^2 + y'^2} + C = 0,$$

that is,
$$Aa^4 + 2\,Ga^2x' + 2\,Fa^2y' + C(x'^2 + y'^2) = 0,$$

which is again the equation of a circle, provided $C \neq 0$. In the special case when $C = 0$, the *given* circle passes through the origin, and its inverse is a straight line. Thus *every circle through the origin is transformed by inversion into a straight line.* It is readily proved conversely that every straight line is transformed into a circle passing through the origin; and in particular that every line through the origin is transformed into itself, as is obvious otherwise.

EXERCISES

1. Find the coordinates of the points inverse to (4, 3), (2, 0), (− 5, 1) with respect to the circle $x^2 + y^2 = 25$.

2. Show that by inversion every line (except a line through the center) is transformed into a circle passing through the center of inversion.

3. Show that all circles with center at the center of inversion are transformed by inversion into concentric circles.

4. Find the equation of the circle about the center of inversion which is transformed into itself.

5. With respect to the circle $x^2 + y^2 = 16$, find the equations of the curves inverse to:

(*a*) $x=5$, (*b*) $x-y=0$, (*c*) $x^2+y^2-6x=0$, (*d*) $x^2+y^2-10y+1=0$, (*e*) $3x-4y+6=0$.

6. Show that the circle $Ax^2 + Ay^2 + 2Gx + 2Fy + a^2A = 0$ is transformed into itself by inversion with respect to the circle $x^2 + y^2 = a^2$.

7. Prove the statements at the end of § 92.

93. Pole and Polar. Let P, P' (Fig. 37) be *inverse* points with respect to the circle (O, a); then the perpendicular l to OP through P' is called the ***polar*** of P, and P the ***pole*** of the line l, with respect to the circle.

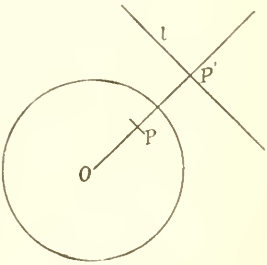


FIG. 37

Notice that (1) if (as in Fig. 37) P lies within the circle, its polar l lies outside; (2) if P lies outside the circle, its polar intersects the circle in two points; (3) if P lies on the circle, its polar is the tangent to the circle at P.

Referring the circle to rectangular axes through its center (Fig. 38) so that its equation is

$$x^2 + y^2 = a^2,$$

we can find the equation of the polar l of any given point $P(x, y)$. For, using as equation of the polar the normal form $X \cos \beta + Y \sin \beta = p$, we have evidently, if P' is the point inverse to P:

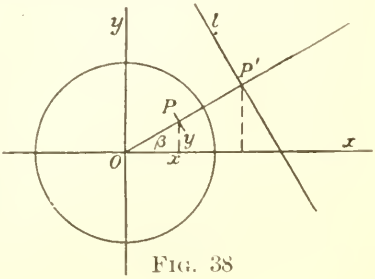


FIG. 38

$$\cos\beta = \frac{x}{\sqrt{x^2+y^2}},\quad \sin\beta = \frac{y}{\sqrt{x^2+y^2}},\quad p = OP' = \frac{a^2}{\sqrt{x^2+y^2}};$$

therefore the equation becomes

$$\frac{xX}{\sqrt{x^2+y^2}} + \frac{yY}{\sqrt{x^2+y^2}} = \frac{a^2}{\sqrt{x^2+y^2}},$$

or simply $$xX + yY = a^2.$$

This then is *the equation of the polar l of the point $P(x, y)$ with respect to the circle of radius a about the origin.* If, in particular, the point $P(x, y)$ lies on the circle, the same equation represents the *tangent to the circle $x^2 + y^2 = a^2$ at the point $P(x, y)$*, as shown previously in § 88.

94. Chord of Contact. *The polar l of any outside point P with respect to a given circle passes through the points of contact C_1, C_2 of the tangents drawn from P to the circle.*

To prove this we have only to show that if C_1 is one of the points of intersection of the polar l of P with the circle, then the angle OC_1P (Fig. 39) is a right angle. Now the triangles OC_1P and $OP'C_1$ are similar since they have the angle at O in common and the including sides proportional owing to the relation

$$OP \cdot OP' = a^2,$$

i.e. $$\frac{OP}{a} = \frac{a}{OP'},$$

where $a = OC_1$. It follows that $\measuredangle OC_1P = OP'C_1 = \frac{1}{2}\pi$.

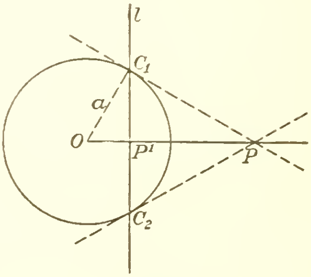


FIG. 39

The rectilinear segment C_1C_2 is sometimes called the *chord of contact* of the point P. We have therefore proved that *the chord of contact of any outside point P lies on the polar of P.*

It follows that *the equations of the tangents that can be drawn from any outside point P to a given circle* can be found by determining the intersections C_1, C_2 of the polar of P with the circle; the tangents are then obtained as the lines joining C_1, C_2 to P.

95. The General Case. The equation of the polar of a point $P(x, y)$ with respect to any circle given in the general form (4), § 80, viz.,

$$(4) \qquad Ax^2 + Ay^2 + 2\,Gx + 2\,Fy + C = 0,$$

is found by the same method that was used in § 89 to generalize the equation of the tangent. Thus, with respect to parallel axes through the center the equation of the circle is

$$x^2 + y^2 = \frac{G^2}{A^2} + \frac{F^2}{A^2} - \frac{C}{A};$$

the equation of the polar of $P(x, y)$ with respect to these axes is by § 93:

$$xX + yY = \frac{G^2}{A^2} + \frac{F^2}{A^2} - \frac{C}{A}.$$

Hence, transferring back to the original axes, we find as *equation of the polar of* $P(x, y)$ *with respect to the circle* (4):

$$AxX + AyY + G(x + X) + F(y + Y) + C = 0.$$

If, in particular, the point $P(x, y)$ lies outside the circle, this polar contains the chord of contact of P; if P lies on the circle, the polar becomes the tangent at P (§ 89).

96. Construction of Polars. *If a point P_1 describes a line l, its polar l_1 with respect to a given circle* (O, a) *turns about a fixed point,* viz., the pole P of the line l (Fig. 40). Conversely, *if a line l_1 turns about one of its points P, its pole P_1 with respect to a given circle* (O, a) *describes a line l, viz. the polar of the point P.*

For, the line l is transformed by inversion with respect to the circle (O, a) into a circle passing through O and through the pole P of l; as this circle must obviously be symmetric with respect to OP it must have OP as diameter. Any point P_1 of l is transformed by inversion into that point Q of the circle of diameter OP at which this circle is intersected by OP_1. The polar of P_1 is the perpendicular through Q to OP_1; it passes therefore through P, wherever P_1 be taken on l.

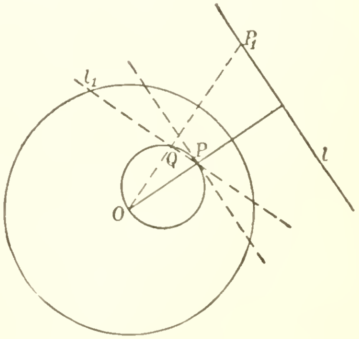


FIG. 40

The proof of the converse theorem is similar.

The pole P_1 of any line l_1 can therefore be constructed as the intersection of the polars of any two points of l_1; this is of advantage when the line l_1 does not meet the circle. And the polar l_1 of any point P_1 can be constructed as the line joining the poles of any two lines through P_1; this is of advantage when the point P_1 lies inside the circle.

EXERCISES

1. Find the equation of the polar of the given point with respect to the given circle and sketch if possible:

(*a*) $(4, 7)$. $x^2 + y^2 = 8$.
(*b*) $(0, 0)$. $x^2 + y^2 - 3x - 4 = 0$.
(*c*) $(2, 1)$. $x^2 + y^2 - 4x - 2y + 1 = 0$.
(*d*) $(2, -3)$. $x^2 + y^2 + 3x + 10y + 2 = 0$.

2. Find the pole of the given line with respect to the given circle and sketch if possible:

(*a*) $x + 2y - 20 = 0$, $x^2 + y^2 = 20$.
(*b*) $x + y + 1 = 0$, $x^2 + y^2 = 4$.
(*c*) $4x - y = 19$, $x^2 + y^2 = 25$.
(*d*) $Ax + By + C = 0$, $x^2 + y^2 = r^2$.
(*e*) $y = mx + b$, $x^2 + y^2 = r^2$.

3. Find the pole of the line joining the points $(20, 0)$ and $(0, 10)$, with respect to the circle $x^2 + y^2 = 25$.

4. Find the tangent to the circle $x^2+y^2-10x+4y+9=0$ at $(7, -6)$.

5. Find the intersection of the tangents to the circle $2x^2 + 2y^2 - 15x + y - 28 = 0$ at the points $(3, 5)$ and $(0, -4)$.

6. Find the tangents to the circle $x^2 + y^2 - 6x - 10y + 2 = 0$ that pass through the point $(3, -3)$.

7. Find the tangents to the circle $x^2 + y^2 - 3x + y - 10 = 0$ that pass through the point $(-\frac{8}{3}, -\frac{14}{3})$.

8. Show that the distances of two points from the center of a circle are proportional to the distances of each from the polar of the other.

9. Show analytically that if two points are given such that the polar of one point passes through the second point, then the polar of the second point passes through the first point.

10. Find the poles of the lines $x - y - 3 = 0$ and $x + y + 8 = 0$ with respect to the circle $x^2 + y^2 - 6x + 4y + 3 = 0$.

97. Power of a Point. If in the left-hand member of the equation of the circle

$$(x-h)^2+(y-k)^2-r^2=0,$$

we substitute for x and y the coordinates x_1, y_1 of a point P_1 not on the circle (Fig. 41), the expression $(x_1-h)^2+(y_1-k)^2-r^2$ is different from zero. Its value is called *the power of the point P_1 (x_1, y_1) with respect to the circle.* As $(x_1-h)^2+(y_1-k)^2$ is the square of the distance $P_1C=d$ between the point P_1 (x_1, y_1) and the center $C(h, k)$, the power of the point P_1 (x_1, y_1) with respect to the circle is d^2-r^2; and this is positive for points without the circle $(d>r)$, zero for points on the circle $(d=r)$, and negative for points within the circle $(d<r)$.

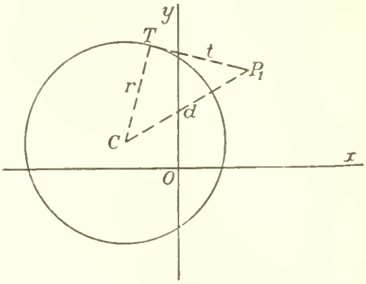


Fig. 41

If the point lies without the circle, its power has a simple interpretation; it is the square of the segment $P_1T=t$ of the tangent drawn from P_1 to the circle:

$$t^2=d^2-r^2=(x_1-h)^2+(y_1-k)^2-r^2.$$

Hence the length t of the tangent that can be drawn from an outside point P_1 (x_1, y_1) to a circle $x^2+y^2+ax+by+c=0$ is given by

$$t^2=x_1^2+y_1^2+ax_1+by_1+c.$$

Notice that the coefficients of x^2 and y^2 must be 1. Compare the similar case of the distance of a point from a line (§ 56).

98. Radical Axis. The locus of a point whose powers with respect to any two circles

$$x^2+y^2+a_1x+b_1y+c_1=0,$$
$$x^2+y^2+a_2x+b_2y+c_2=0,$$

are equal is given by the equation

$$x^2+y^2+a_1x+b_1y+c_1=x^2+y^2+a_2x+b_2y+c_2,$$

which reduces to

$$(a_1-a_2)x+(b_1-b_2)y+(c_1-c_2)=0.$$

This locus is therefore a straight line; it is called the ***radical axis*** of the two circles. It always exists unless $a_1=a_2$ and $b_1=b_2$, *i.e.* unless the circles are concentric.

Three circles taken in pairs have three radical axes which pass through a common point, called the ***radical center***. For, if the equation of the third circle is

$$x^2 + y^2 + a_3x + b_3y + c_3 = 0,$$

the equations of the radical axes will be

$$(a_2 - a_3)x + (b_2 - b_3)y + (c_2 - c_3) = 0,$$
$$(a_3 - a_1)x + (b_3 - b_1)y + (c_3 - c_1) = 0,$$
$$(a_1 - a_2)x + (b_1 - b_2)y + (c_1 - c_2) = 0.$$

These lines intersect in a point, since the determinant of the coefficients in these equations is equal to zero (Ex. 10, p. 57).

99. Family of Circles. The equation

$$(8) \quad (x^2 + y^2 + a_1x + b_1y + c_1) + \kappa(x^2 + y^2 + a_2x + b_2y + c_2) = 0$$

represents *a family, or pencil, of circles each of which passes through the points of intersection of the circles*

$$(9) \qquad x^2 + y^2 + a_1x + b_1y + c_1 = 0,$$

and

$$(10) \qquad x^2 + y^2 + a_2x + b_2y + c_2 = 0,$$

if these circles intersect. For, the equation (8) written in the form

$$(1 + \kappa)x^2 + (1 + \kappa)y^2 + (a_1 + \kappa a_2)x + (b_1 + \kappa b_2)y + c_1 + \kappa c_2 = 0$$

represents a circle for every value of κ except $\kappa = -1$, as the coefficients of x^2 and y^2 are equal and there is no xy-term (§ 79). Each one of the circles (8) passes through the common points of the circles (9) and (10) if they have any, since the equation (8) is satisfied by the coordinates of those points which satisfy both (9) and (10). Compare § 58. The constant κ is called the *parameter* of the family.

In the special case when $\kappa = -1$, the equation is of the first degree and hence represents a line, viz. the radical axis (§ 98) of the two circles (9), (10). If the circles intersect, the radical axis contains their *common chord.*

EXERCISES

1. Find the powers of the following points with respect to the circle $x^2 + y^2 - 3x - 2y = 0$ and thus determine their positions relative to the circle: $(2, 0)$, $(0, 0)$, $(0, -4)$, $(3, 2)$.

2. What is the length of the tangent to the circle: (*a*) $x^2 + y^2 + ax + by + c = 0$ from the point $(0, 0)$, (*b*) $(x-2)^2 + (y-3)^2 - 1 = 0$ from the point $(4, 4)$?

3. By § 97, $t^2 = d^2 - r^2 = (d+r)(d-r)$; interpret this relation geometrically.

4. Find the radical axis of the circles $x^2 + y^2 + ax + by + c = 0$ and $x^2 + y^2 + bx + ay + c = 0$ and the length of the common chord.

5. Find the radical center of the circles $x^2 + y^2 - 3x + 4y - 7 = 0$, $x^2 + y^2 = 16$, $2(x^2 + y^2) + 6x + 1 = 0$. Sketch the circles and their radical axes.

6. Find the circle that passes through the intersections of the circles $x^2 + y^2 + 5x = 0$ and $x^2 + y^2 + x - 2y - 5 = 0$, and (*a*) passes through the point $(-5, 6)$, (*b*) has its center on the line $4x - 2y - 15 = 0$, (*c*) has the radius 5.

7. Sketch the family of circles $x^2 + y^2 - 6y + \kappa(x^2 + y^2 + 3y) = 0$.

8. What family of circles does the equation $Ax + By + C + \kappa(x^2 + y^2 + ax + by + c) = 0$ represent?

9. Find the family of curves inverse to the family of lines $y = mx + b$; (*a*) with m constant and b variable, (*b*) with m variable and b constant. Draw sketches for each case.

10. Show that a circle can be drawn orthogonal to three circles, provided their centers are not in a straight line.

11. Find the locus of a point whose power with respect to the circle $2x^2 + 2y^2 - 5x + 11y - 6 = 0$ is equal to the square of its distance from the origin. Sketch.

12. Show that the locus of a point for which the sum of the squares of its distances from the four sides of a square is constant, is a circle. For what value of the constant is the circle real? For what value is it the inscribed circle?

13. Find the locus of a point if the sum of the squares of its distances from the sides of an equilateral triangle of side $2a$ is constant.

14. Show that the circle through the points $(2, 4)$, $(-1, 2)$, $(3, 0)$ is orthogonal to the circle which is the locus of a point the ratio of whose distances from the points $(2, 4)$ and $(-1, 2)$ is 3. Sketch.

15. Show that the circles through two fixed points, say $(-a, 0)$, $(a, 0)$, form a family like that of Ex. 8.

16. The locus of a point whose distances from the fixed points $(-a, 0)$, $(a, 0)$ are in the constant ratio κ $(\neq 1)$ is the circle

$$x^2 + y^2 + 2\frac{1+\kappa^2}{1-\kappa^2}ax + a^2 = 0.$$

Compare Ex. 9, p. 90. Show that, whatever κ $(\neq 1)$, this circle intersects every circle of the family of Ex. 15 at right angles.

Parameters. In problems on loci it is often convenient to express the coordinates x, y of the point describing the locus in terms of a third variable and then to eliminate this variable. Thus, for any point on a circle of radius a about the origin we have evidently

$$(a) \qquad x = a\cos\phi, \qquad y = a\sin\phi;$$

eliminating ϕ by squaring and adding we find

$$x^2 + y^2 = a^2.$$

The variable ϕ is called the *parameter;* the equations (a) are the *parameter equations* of a circle about the origin.

17. The ends A, B of a straight rod of length $2a$ move along two perpendicular lines; find the locus of the midpoint of AB.

18. One end A of a straight rod of length a describes a circle of radius a and center O, while the other end B moves along a line through O. Taking this line as axis Ox and O as origin, find the locus of the intersection of OA (produced) with the perpendicular to the axis Ox through B.

19. Four rods are jointed so as to form a parallelogram; if one side is fixed, find the path described by any point rigidly connected with the opposite side.

20. An *inversor* is any mechanism for describing the inverse of a given curve. Peaucellier's cell consists of a linked rhombus $APBP'$ attached by means of two equal links OA, OB to a fixed point O. Show that this linkage is an inversor, with O as center.

CHAPTER VII

COMPLEX NUMBERS

PART I. THE VARIOUS KINDS OF NUMBERS

100. Introduction. The process of finding the points of intersection of a line and a circle (§ 86) involves the solution of a quadratic equation. The solution of such a quadratic equation may involve the square root of a negative number. Thus the roots of $x^2 - 2x + 3 = 0$ are $x = 1 \pm \sqrt{-2}$.

The square root, or in fact any even root, of a negative number is called an *imaginary number*; and an expression of the form $a + \sqrt{-b}$ in which a is any real number and b any positive real number is called a ***complex number***.

We shall first recall briefly the successive steps by which, in elementary algebra, we are led from the positive integers to other kinds of numbers.

101. Fundamental Laws of Algebra. The so-called *natural numbers*, or ***positive integers*** 1, 2, 3, 4, ⋯ form a class of things for which the operations of *addition* and *multiplication* have a clear and well-known meaning. These operations are governed by the following laws:

(*a*) the ***commutative law*** for addition and for multiplication:

$$a + b = b + a, \qquad ab = ba:$$

(*b*) the ***associative law*** for addition and for multiplication:

$$(a + b) + c = a + (b + c), \qquad (ab)c = a(bc):$$

(*c*) the ***distributive law***, connecting addition and multiplication:

$$(a + b)c = ac + bc, \qquad a(b + c) = ab + ac.$$

102. Inverse Operations. The result obtained by adding or multiplying any two or more positive integers is always again a positive integer.

This is not true for the so-called *inverse operations: subtraction*, the inverse of addition, and *division*, the inverse of multiplication. To make these inverse operations always possible the domain of positive integers is extended by introducing:

(*a*) the negative numbers and the number zero;

(*b*) the (positive and negative) rational fractions.

The relation between these various kinds of numbers is best understood by imagining the positive integers represented by equidistant points on a line, or rather by the distances of these points from a common origin O (Fig. 42).

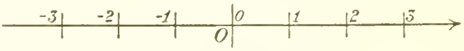


Fig. 42

Negative numbers are then represented by equidistant points on the opposite side of the origin; zero is represented by the origin; and fractions correspond to intermediate points.

103. Rational Numbers. The positive and negative integers, the rational fractions, and zero, form the domain of ***rational numbers.*** By adopting the well-known *rules of signs* the operations of addition and multiplication and their inverses, subtraction and division, can be extended to these rational numbers; and all four of these operations, *with the single exception of division by zero*, can be shown to be always possible in the domain of rational numbers, so that any finite number of such operations performed with a finite number of rational numbers produces again a rational number.

In the domain of positive integers such linear equations as $x+7=0$, $5x-3=0$ cannot be solved. But in the domain of rational numbers the linear equation $ax+b=0$ can always be solved if a and b are rational and a is not zero.

104. Laws of Exponents. In the domain of positive integers, we pass from addition to multiplication by denoting a sum of b terms each equal to a by the symbol ab, called the product of a and b. Similarly, we may denote a product of b factors each equal to a by the symbol a^b; this operation is called *raising a to the bth power*, or *involution*. By this definition, the symbol a^b has a meaning only when the exponent b is a positive integer. But the base a may evidently be any rational number. The ***laws of exponents***, or of indices,

$$a^p \cdot a^q = a^{p+q}, \quad a^p \cdot b^p = (ab)^p, \quad (a^p)^n = a^{pn},$$

follow directly from the definition of the symbol a^b. The result of raising any rational number to a positive integral power is always a rational number.

105. The Inverses of Involution. It should be observed that the symbol a^b differs from the symbols $a + b$ and ab in not being commutative (§ 101); *i.e.* in general a and b cannot be interchanged:

$$a^b \neq b^a, \quad \text{if} \quad b \neq a.$$

It follows from this fact that while addition and multiplication have each but one inverse operation, involution has two:

(a) If in the relation

$$a^b = c$$

b and c are regarded as known, the operation of finding a is called *extracting the bth root of c*, or *evolution*, and is expressed in the form

$$a = \sqrt[b]{c}.$$

(b) If in the same relation a and c are regarded as known, the operation of finding b is called *taking the logarithm of c to the base a* and is indicated by

$$b = \log_a c.$$

Logarithms will be discussed in Chapter XII; for the present we shall consider only the former inverse operation.

106. Irrational Numbers. Even when a, b, and therefore c are positive integers, the extraction of roots is often impossible, not only in the domain of positive integers, but even in the domain of rational numbers. Thus, in so simple a case as $b = 2$, $c = 2$, we find that $a = \sqrt{2}$ is not a rational number, *i.e.* it is not the quotient of any two integers, however large. For, suppose that $\sqrt{2} = h/k$, where h and k are integers and the rational fraction h/k is reduced to its lowest terms; then squaring both sides, we find $2 = h^2/k^2$. But the rational fraction h^2/k^2 is also reduced to its lowest terms and consequently cannot be equal to the integer 2.

We are thus led to a new extension of the number system by including the results of evolution: any root of a rational number that is not a rational number is called an ***irrational number.*** The rational and irrational numbers together form the domain of ***real numbers.***

If numbers are represented by points on a line as in § 102, the number $\sqrt{2}$ has a single definite point corresponding to it on the line; for, the segment representing it can be found as the hypotenuse of a right triangle whose sides have the length 1. It can be shown that a single definite point corresponds to any given irrational number.

It thus appears that although the rational numbers, "crowd the line," *i.e.* although between any two rational numbers, however close, we can insert other rational numbers, they do not "fill" the line; *i.e.* there are points on the line that cannot be represented exactly by rational numbers.

107. Extension of Laws. A rigorous definition and discussion of irrational numbers requires somewhat long and complicated developments. It will here suffice to state the result that *irrational numbers are subject to the same rules of operation as are rational numbers.*

The fundamental laws of addition and multiplication (§ 101) hold therefore for all real numbers, and so do the laws of signs of elementary algebra. As regards the laws of exponents (§ 104), they can be shown to hold when the bases are any real numbers. Moreover, it can be shown that the symbol a^b has a definite meaning even when the exponent b is any real number, and that the laws of exponents hold for such powers, provided only that the bases are positive. It is known from elementary algebra how this can be done for *rational* exponents by defining the symbols a^0 and a^{-m} as

$$a^0 = 1, \quad a^{-m} = \frac{1}{a^m};$$

and it is shown in the theory of irrational numbers that the latter definition can be used even when m is irrational.

Thus the laws of exponents (§ 104) hold for any *real* exponents provided the bases are positive.

108. Measurement. Historically, the gradual introduction of rational fractions, of negative numbers, of irrational numbers, was determined very largely by the *applications* of arithmetic and algebra. Any magnitude that can be subdivided indefinitely into parts of the same kind as the whole, and hence can be "measured," leads naturally to the idea of the fraction. Magnitudes that can be measured in two opposite senses, like the distance along a line, the height of the thermometer above and below the zero point, credit and debit, the height of the water level above or below a fixed point, suggest the idea of negative numbers. The incommensurable magnitudes that occur frequently in geometry lead to the introduction of irrational numbers. One of the principal advantages of algebra consists in the remarkable fact that all these different kinds of numbers are subject to the same simple laws of operation.

109. Imaginary Numbers. As mentioned in § 107, there is still a restriction, in the domain of real (*i.e.* rational and irrational) numbers, to the use of the laws of exponents (§ 104): the square root of a negative number has no meaning in this domain.

Thus, $\sqrt{-2}$ is not a real number; for, by the definition of the square root, the square of $\sqrt{-2}$ is -2; but there exists no real number whose square is -2. In other words, such simple equations as $x^2+2=0$, $x^2-2x+3=0$ have no real solutions. It has therefore been found of advantage to give one further extension to the meaning of the term "number," by including the even roots of negative numbers, under the name of *imaginary numbers.*

110. The Imaginary Unit. Any even root of a negative (rational or irrational) number is defined as an ***imaginary number.*** Every such number can be reduced to the form $\pm\sqrt{-a}$, where a is positive. It is customary to denote $\sqrt{-1}$ by the letter i and call it the ***imaginary unit.*** Any imaginary number $\pm\sqrt{-a}$ can therefore be written in the form

$$\pm\sqrt{-a} = \pm\sqrt{a}\, i;$$

that is, every imaginary number is a real multiple of the imaginary unit i. Notice that as $i=\sqrt{-1}$ we always have

$$i^2 = -1.$$

The algebraic sum of a real number and an imaginary number, *i.e.* the expression $a+bi$ where a and b are real, is called a ***complex number.*** Notice that the domain of complex numbers includes both real and imaginary numbers. For, the complex number $a+bi$ is real in the particular case when $b=0$, it is an imaginary number if $a=0$. The great advantage of complex numbers lies in the fact that all the seven fundamental operations of algebra (viz. addition, subtraction, multiplication, division, involution, evolution, and logarithmization), with the single exception of division by zero, can be performed on complex numbers, the result being always a complex number; *i.e.* if we denote by α, β any two complex numbers, then $\alpha + \beta$, $\alpha - \beta$, $\alpha\beta$, α / β, α^β, $\sqrt[\beta]{\alpha}$, $\log_\beta \alpha$ can all be expressed in the form $a + bi$. It can then be shown that every algebraic equation of the nth degree has n complex roots.

111. Imaginary Values in Analytic Geometry. In elementary analytic geometry we are concerned with "real" points and lines, *i.e.* with points whose coordinates are real and with lines whose equations have real coefficients. But it should be observed that points with complex coordinates may lie on real lines and that lines with complex coefficients may contain real points. Thus, the coordinates of the point $(2 + 3i, 1 - 2i)$ satisfy the equation of the real line $2x + 3y - 7 = 0$, and the equation $(1 + 2i)x - (2 + 3i)y + 1 = 0$ is satisfied by the point $(3, 2)$. Calculations with imaginary points and lines may therefore lead to results about real points and lines.

A rather striking example is afforded by the the theory of poles and polars with respect to the circle. We have seen (§§ 93–95) that with respect to a given circle every line of the plane (excepting those through the center) has a real pole and every point (excepting the center) has a real polar. If the line l intersects the circle in two points Q_1, Q_2, its pole P can be found as the intersection of the tangents at Q_1, Q_2. If the line l does not intersect the circle, this geometrical construction is impossible. But the analytic process of finding the points of intersection of the line l with the circle can be carried through. The coordinates of the points of intersection will be imaginary; and hence the equations of the tangents at these points will have imaginary coefficients. But the point of intersection of these imaginary lines will be a real point; viz. the pole P of the line l and its real coordinates can be found in this way.

Thus to find the pole of the line $y = 2$ with respect to the circle $x^2 + y^2 = 1$ we obtain the imaginary points of intersection $(\sqrt{3}\,i, 2)$ and $(-\sqrt{3}\,i, 2)$; the imaginary tangents at these points are therefore: $\sqrt{3}\,ix + 2y = 1$, $-\sqrt{3}\,ix + 2y = 1$; these imaginary lines intersect in the real point $(0, \frac{1}{2})$; it is easy to show that this is the required pole.

PART II. GEOMETRIC INTERPRETATION OF COMPLEX NUMBERS

112. Representation of Imaginaries. The meaning of complex numbers will best be understood from their graphical representation.

We have seen (§ 102) that every real number a can be represented by a point A on a straight line on which an origin O and a positive sense have been selected. We shall call this line (Fig. 43) the *axis of real numbers*, or briefly the *real axis*.

To represent the imaginary numbers we draw an axis through O at right angles to the real axis and call it the *axis of imaginary numbers*, or briefly the *imaginary axis*. The point A' on this axis, at the distance $OA' = a$ from the origin, can then be taken as representing the imaginary number ai.

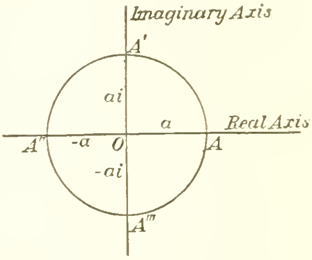


FIG. 43

113. Representation by Rotation. This representation is also suggested by the fundamental rule for dealing with imaginary numbers that $i^2 = -1$. For, if a be any real number and A its representative point on the real axis, the real number $-a$ has its representative point A'' situated symmetrically to A with respect to O on the real axis: in other words, the segment OA'' which represents $-a$ can be regarded as obtained from the segment OA that represents a by turning OA through two right angles about O. Thus the factor $-1 = i^2$ applied to the number a, or rather to the segment OA, turns it about O through two right angles. This suggests the idea that the factor $\sqrt{-1} = i$, applied to a, may be interpreted as turning the segment OA *through one right angle in the counterclockwise sense* so as to make it take the position OA'. Indeed, if the factor i be now applied to ai, *i.e.* to the segment OA', it will turn OA' into OA'' and produce $ai^2 = -a$.

Turning OA'' counterclockwise through a right angle, we obtain the point A''' on the imaginary axis which represents $ai^3 = -ai$; and finally, turning OA''' counterclockwise through a right angle we regain the starting point A which represents $ai^4 = a$.

114. Representation of Complex Numbers. A *complex number, i.e.* an expression of the form

$$z = x + yi,$$

where x, y are real numbers while i is the imaginary unit $\sqrt{-1}$, is fully determined by the two real numbers x and y, provided we know which of these is to be the real part. If we take the real axis as axis Ox, the imaginary axis as axis Oy, of a rectangular coordinate system (Fig. 44), the numbers x, y determine a definite point of the plane, and only one. This point $P(x, y)$ can therefore be taken as representative of the complex number $z = x + yi$.

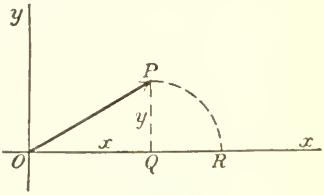


FIG. 44

This representation also agrees with the idea (§ 113) that the factor i turns through a right angle. For if we lay off on the real axis, or axis Ox, $OQ = x$, and on the same axis $QR = y$ we obtain $OR = OQ + QR = x + y$; and if we turn QR about Q through a right angle into QP we obtain $x + yi$ and reach the point P.

To every complex number $z = x + yi$ thus corresponds one and only one point $P(x, y)$; to every point $P(x, y)$ of the plane corresponds one and only one complex number $z = x + yi$.

The real numbers, and only these, have their representative points on the axis Ox; the imaginary numbers have theirs on the axis Oy. The origin $(0, 0)$ represents the complex number $0 + i0 = 0$.

115. Correspondence of Complex Numbers to Vectors. It should be recalled that strictly speaking (§ 102) a real number x is represented, not by a point A of the real axis, but by the segment $OA = x$. Similarly the complex number $z = x + yi$ is represented, strictly speaking, not by the point P (Fig. 44), but rather by the radius vector OP, taken with a definite direction and sense. Thus the complex number $z = x + yi$ represents a *vector* (see §§ 19–20), whose rectangular components are x and y. It will be shown below that the addition and subtraction of complex numbers follow exactly the laws of the composition of (concurrent) forces, velocities, translations, etc., in the same plane.

116. Equality of Complex Numbers. *Two complex numbers* $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$ *are called* **equal**, *if, and only if, their representative points coincide*, i.e. $z_1 = z_2$ *if*

$$x_1 = x_2 \text{ and } y_1 = y_2,$$

just as two forces are equal only when their rectangular components are equal respectively.

If we apply the ordinary rules of algebra to the equation

$$x_1 + y_1 i = x_2 + y_2 i$$

we obtain

$$x_1 - x_2 = (y_2 - y_1)i.$$

Now the real number $x_1 - x_2$ cannot be equal to the imaginary number $(y_2 - y_1)i$ unless $x_1 - x_2 = 0$ and $y_2 - y_1 = 0$; whence again we find $x_1 = x_2$, $y_1 = y_2$.

It follows in particular that the complex number $z = x + yi$ is zero if, and only if, $x = 0$ and $y = 0$.

EXERCISES

1. Locate the points which represent the following complex numbers:

(*a*) $4 - 3i$. (*b*) $2i$. (*c*) $-1 - i$. (*d*) 4.
(*e*) $.4 + .1i$. (*f*) $\frac{2}{3} - \frac{1}{2}i$. (*g*) $-10 - \frac{7}{3}i$. (*h*) $-\frac{4}{5}i$.

2. Find the values of m and n in the following equations:

(*a*) $(m - n) + (m + n - 2)i = 0$. (*b*) $(m^2 + n^2 - 25) + (m - n - 1)i = 0$.
(*c*) $m + ni = 3 - 2i$. (*d*) $mni = m^2 - n^2 + 4i$.

3. Show that

(*a*) $i^3 = -i$. (*b*) $i^5 = i^9$. (*c*) $i^6 + i^8 = 0$. (*d*) $i^4 - i^6 = 2$.

4. Show that the following relations are true, n being any positive integer:

(*a*) $i^{4n} = 1$. (*b*) $i^{4n+3} = -i$. (*c*) $i^{4n} - i^{4n+2} = 2$.

5. Show that

(*a*) $\frac{1}{2}(-1 + \sqrt{3}\,i)$ is a cube root of 1,
(*b*) $\frac{1}{2}(+1 - \sqrt{3}\,i)$ is a cube root of -1.

117. Addition of Complex Numbers. *The sum of two complex numbers* $z_1 = x_1 + y_1 i$ *and* $z_2 = x_2 + y_2 i$ *is defined as the complex number* $z = (x_1 + x_2) + (y_1 + y_2)i$: in other words, if (Fig. 45) P_1 is the point that represents z_1 and P_2 the point that represents z_2, then the point P that represents the sum $z = z_1 + z_2$ has for its abscissa the sum of the abscissas of P_1 and P_2 and for its ordinate the sum of the ordinates of P_1 and P_2. It appears from the figure that this point P is the fourth vertex of the parallelogram of which the other three vertices are the origin O and the points P_1, P_2.

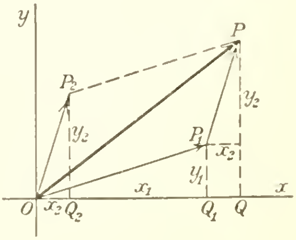


FIG. 45

118. Analogy to Parallelogram Law of Vectors. By comparing §§ 19, 20 it will be clear that the addition of two complex numbers consists in finding the resultant OP of their representative vectors OP_1, OP_2. The vectors may be thought of as forces, velocities, translations, etc. In the case of translations this composition of two successive translations into a single equivalent translation is particularly obvious.

While a real number $x = OQ$ represents a translation along the axis Ox, an imaginary number yi a translation along the axis Oy, a complex number $z = x + yi$ can be interpreted as representing a translation OP in any direction (Fig. 44). The succession of two such translations $z_1 = x_1 + y_1 i$ represented by OP_1 (Fig. 45) and $z_2 = x_2 + y_2 i$ represented by OP_2 is equivalent to the single translation $z = (x_1 + x_2) + (y_1 + y_2)i$ represented by OP.

It follows that the addition of any number of complex numbers (Fig. 46) whose representative vectors are OP_1, OP_2, OP_3, OP_4 can be effected by forming the polygon $OP_1P_2'P_3'P$: the closing line OP is the representative vector of the sum; precisely as in finding the resultant of concurrent forces (§ 20).

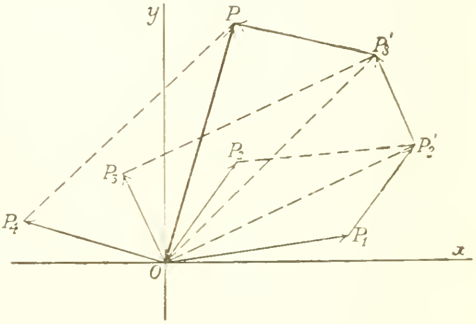


FIG. 46

119. Subtraction. *The difference of two complex numbers* $z_1 = x_1 + y_1 i$ *and* $z_2 = x_2 + y_2 i$ *is defined as the complex number* $z = (x_1 - x_2) + (y_1 - y_2)i$. Its representative point P is found geometrically by laying off from P_1 (Fig. 47) a segment P_1P equal and opposite to OP_2, *i.e.* equal and parallel to P_2O.

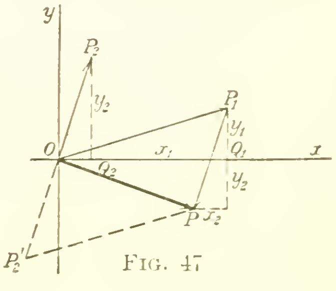


FIG. 47

120. Multiplication. *The **product** of two complex numbers $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$ is found by multiplying these two expressions according to the ordinary rules of algebra and observing that $i^2 = -1$.* We thus find:

$$\begin{aligned} z_1 z_2 = (x_1 + y_1 i)(x_2 + y_2 i) &= x_1 x_2 + x_1 y_2 i + x_2 y_1 i + y_1 y_2 i^2 \\ &= (x_1 x_2 - y_1 y_2) + (x_1 y_2 + x_2 y_1) i, \end{aligned}$$

which is a complex number. A geometric construction will be given in § 124.

121. Conjugate Imaginaries. Two complex numbers that differ only in the sign of the imaginary part are called **conjugate** complex numbers. Thus, the conjugate of $5 + 2i$ is $5 - 2i$; that of $-3 - i$ is $-3 + i$, etc. The radii vectores representing two conjugate numbers are situated symmetrically with respect to the real axis.

The product of two conjugate complex numbers is a real number; for

$$(x + yi)(x - yi) = x^2 + y^2.$$

Notice that the roots of a quadratic equation are conjugate complex numbers.

122. Division. To form the **quotient** of two complex numbers we may *render the denominator real*, by multiplying both numerator and denominator by the conjugate of the denominator. Thus:

$$\begin{aligned} \frac{z_1}{z_2} = \frac{x_1 + y_1 i}{x_2 + y_2 i} = \frac{(x_1 + y_1 i)(x_2 - y_2 i)}{(x_2 + y_2 i)(x_2 - y_2 i)} &= \frac{x_1 x_2 - x_1 y_2 i + x_2 y_1 i + y_1 y_2}{x_2^2 + y_2^2}, \\ &= \left(\frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2}\right) + \left(\frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2}\right) i. \end{aligned}$$

Here also the result is a complex number. A geometric construction is indicated in § 125.

EXERCISES

1. Simplify the following expressions and illustrate by geometric construction :

(*a*) $(3-6i)+(4-2i)$. (*b*) $(4-3i)-(2+i)$.
(*c*) $(6+i)+(3-2i)-(i)$. (*d*) $(2-3i)-(-1+i)-(3+5i)$.
(*e*) $(4)-(3i)$. (*f*) $(i)+(3-2i)-(5)$.

2. Write the following products as complex numbers and locate the corresponding points :

(*a*) $(\sqrt{5}+i\sqrt{6})(\sqrt{6}+i\sqrt{5})$. (*b*) $(3-i\sqrt{8})(\sqrt{3}+i\sqrt{2})$.
(*c*) $(\sqrt{1+i}-\sqrt{1-i})^2$. (*d*) $(\sqrt{a}-\sqrt{-a})^3$.

3. Show that

(*a*) $\dfrac{1+2i}{1+i}+\dfrac{1-2i}{1-i}=3$. (*b*) $(x+yi)^2-(x-yi)^2=4xyi$.
(*c*) $(x+yi)^4+(x-yi)^4=2(x^4+y^4)-12x^2y^2$.

4. Write the following quotients as complex numbers and locate the corresponding points :

(*a*) $\dfrac{2+3i}{4-i}$. (*b*) $\dfrac{1+i}{i}$. (*c*) $\dfrac{5-3i}{5+3i}$.
(*d*) $\dfrac{(1+i)(1+2i)(1+3i)}{1+4i}$. (*e*) $\dfrac{1}{-7+2i}$. (*f*) $\dfrac{-1}{3-4i}$.

5. Verify by geometric construction that the sum of two conjugate complex numbers is a real number and that the difference is an imaginary number.

6. Evaluate the following expressions for $z_1=3+4i$ and $z_2=-2+5i$ and check by geometric construction :

(*a*) z_1-6. (*b*) $2z_2+3$. (*c*) $6-5z_1$. (*d*) $3i+2z_1$.
(*e*) $2i-\frac{1}{2}z_1$. (*f*) $2-2z_2$. (*g*) $\frac{2}{3}(i-z_1)$. (*h*) $-3i-z_2$.
(*i*) z_1+2z_2. (*j*) $3z_1+z_2$. (*k*) z_1-2z_2. (*l*) $z_2-\frac{1}{2}z_1$.
(*m*) z_1+5z_2-4i. (*n*) $z_2-\frac{1}{3}z_1+3$. (*o*) $5-z_1-z_2$. (*p*) $z_2-6-\frac{3}{4}z_1$.

7. Let X_1 and Y_1 represent the projections of a force F_1 on the axes of x and y, respectively, and X_2 and Y_2 those of a second force F_2. Show, by the parallelogram law, that the projections on the axes of the *resultant* (or sum) of F_1 and F_2 are X_1+X_2 and Y_1+Y_2.

8. From Ex. 7, show that the correct results are obtained if F_1 is represented by X_1+Y_1i, F_2 by X_2+Y_2i, and their resultant by

$$F_1+F_2=(X_1+Y_1i)+(X_2+Y_2i)=(X_1+X_2)+(Y_1+Y_2)i.$$

123. Polar Representation. The use of the polar coordinates r, ϕ of the representative point $P(x, y)$ leads to simple interpretations of multiplication, division, involution, and evolution.

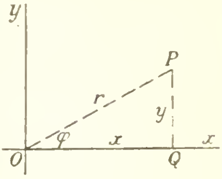


FIG. 48

The distance $OP = r$ (Fig. 48) is called the ***modulus*** or *absolute value* of the complex number; the vectorial angle ϕ is sometimes called the *argument*, *phase*, or ***amplitude***.

Since $\quad x = r\cos\phi$ and $y = r\sin\phi$,

we can write $\quad z = x + yi = r(\cos\phi + i\sin\phi)$.

The right-hand member of this equation is the ***polar form*** of the complex number $z = x + yi$.

124. Products in Polar Form. The product of two complex numbers $z_1 = r_1(\cos\phi_1 + i\sin\phi_1)$ and $z_2 = r_2(\cos\phi_2 + i\sin\phi_2)$ is

$$\begin{aligned} z_1z_2 &= r_1(\cos\phi_1 + i\sin\phi_1)r_2(\cos\phi_2 + i\sin\phi_2) \\ &= r_1r_2[(\cos\phi_1\cos\phi_2 - \sin\phi_1\sin\phi_2) + i(\sin\phi_1\cos\phi_2 + \cos\phi_1\sin\phi_2)] \\ &= r_1r_2[\cos(\phi_1+\phi_2) + i\sin(\phi_1+\phi_2)]. \end{aligned}$$

This shows that *the modulus of the product of two complex numbers is the product of the moduli, the amplitude of the product is the sum of the amplitudes, of the factors.*

The point P that represents the product of the complex numbers represented by the points P_1 and P_2 (Fig. 49) can be constructed as follows: Let P_0 be the point on the axis Ox at unit distance from the origin O and draw the triangle OP_0P_1; on OP_2 construct the similar triangle OP_2P. The point P thus located is the required point. For, by construction the angle $P_2OP = \phi_1$, hence the angle $P_0OP = \phi_1 + \phi_2$. Moreover, as the triangles OP_0P_1 and OP_2P are similar, their sides are proportional, *i.e.*

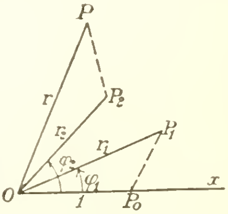


FIG. 49

$$1 : r_1 = r_2 : OP, \text{ whence } OP = r_1r_2.$$

125. Quotients in Polar Form. For the quotient of the two complex numbers $z_1 = r_1(\cos\phi_1 + i\sin\phi_1)$ and $z_2 = r_2(\cos\phi_2 + i\sin\phi_2)$ we find by making the denominator real:

$$\frac{z_1}{z_2}=\frac{r_1(\cos\phi_1+i\sin\phi_1)}{r_2(\cos\phi_2+i\sin\phi_2)}=\frac{r_1(\cos\phi_1+i\sin\phi_1)(\cos\phi_2-i\sin\phi_2)}{r_2(\cos\phi_2+i\sin\phi_2)(\cos\phi_2-i\sin\phi_2)}$$

$$=\frac{r_1}{r_2}\cdot\frac{(\cos\phi_1\cos\phi_2+\sin\phi_1\sin\phi_2)+i(\sin\phi_1\cos\phi_2-\cos\phi_1\sin\phi_2)}{\cos^2\phi_2+\sin^2\phi_2}$$

$$=\frac{r_1}{r_2}[\cos(\phi_1-\phi_2)+i\sin(\phi_1-\phi_2)].$$

Hence *the modulus of the quotient* $z = z_1/z_2$ *is the quotient of the moduli, the amplitude is the difference of the amplitudes of* z_1 *and* z_2. Evidently the point P that represents the quotient $z = z_1/z_2$ (Fig. 50) can be located by reversing the geometric construction given in § 124; *i.e.* by constructing on the unit segment OP_0 the triangle OP_0P similar to the triangle OP_2P_1.

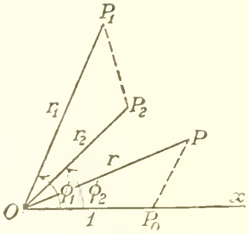


FIG. 50

EXERCISES

1. Write the following complex numbers in polar form:

(*a*) $2+2\sqrt{3}\,i$. (*b*) $-3+3\sqrt{3}\,i$. (*c*) $6-6\,i$. (*d*) $-5\,i$.

(*e*) 7. (*f*) -8. (*g*) $5\sqrt{3}-5\,i$. (*h*) $-10-10\,i$.

2. Write the following complex numbers in the form $x+yi$:

(*a*) $3(\cos 30° + i\sin 30°)$. (*b*) $5(\cos\frac{1}{3}\pi + i\sin\frac{1}{3}\pi)$.

(*c*) $10(\cos\frac{3}{2}\pi + i\sin\frac{3}{2}\pi)$. (*d*) $4(\cos\frac{5}{4}\pi + i\sin\frac{5}{4}\pi)$.

(*e*) $\sqrt{2}(\cos\frac{1}{4}\pi + i\sin\frac{1}{4}\pi)$. (*f*) $\sqrt{3}(\cos\frac{4}{3}\pi + i\sin\frac{4}{3}\pi)$.

(*g*) $7(\cos 0 + i\sin 0)$. (*h*) $5(\cos\pi + i\sin\pi)$.

(*i*) $2\sqrt{3}(\cos\frac{5}{6}\pi + i\sin\frac{5}{6}\pi)$. (*j*) $5\sqrt{2}(\cos\frac{7}{4}\pi + i\sin\frac{7}{4}\pi)$.

(*k*) $11(\cos\frac{1}{2}\pi + i\sin\frac{1}{2}\pi)$. (*l*) $8(\cos 75° + i\sin 75°)$.

3. Put the following complex numbers in polar form, perform the indicated multiplication or division, and write the result in the form $x+yi$. Check by algebra and illustrate by geometry.

(*a*) $(2\sqrt{3}+2\,i)(3+3\sqrt{3}\,i)$. (*b*) $(1+i)(2+2\,i)$.

(*c*) $(-2-2\,i)(5+5\,i)$. (*d*) $(-4+4\sqrt{3}\,i)(-3-3\sqrt{3}\,i)$.

(*e*) $(1+\sqrt{3}\,i)(1-\sqrt{3}\,i)$. (*f*) $(-2)(-3\,i)$.

(*g*) $\dfrac{2\sqrt{3}-2\,i}{-5\,i}$. (*h*) $\dfrac{4\,i}{5+5\,i}$. (*i*) $\dfrac{-7}{-3+3\sqrt{3}\,i}$.

(*j*) $\dfrac{1+i}{1-i}$. (*k*) $\dfrac{1}{-\sqrt{3}-i}$. (*l*) $\dfrac{-i}{i}$.

4. Show that the modulus of the product of the complex numbers $a + bi$ and $c + di$ is $\sqrt{(a^2 + b^2)(c^2 + d^2)}$.

5. Show by geometric construction that the product of two conjugate complex numbers is a real number.

6. Show how to locate by geometric construction the point which represents the reciprocal of a complex number.

7. Show that the point P that represents a complex number z and the point P' that represents the conjugate of the reciprocal $1/z$ are inverse points with respect to the unit circle about the origin.

8. With respect to the unit circle about the origin, find the complex numbers representing the points inverse to

(*a*) $3 + 4i$. (*b*) $3 + \sqrt{-5}$. (*c*) $-5 + 3i$. (*d*) $1 - 6i$.

9. Show that the ratio of two complex numbers whose amplitudes differ by $\pm \frac{1}{2}\pi$ is an imaginary number.

10. Show that the ratio of two complex numbers whose amplitudes are equal or differ by $\pm \pi$ is a real number.

126. De Moivre's Theorem. The rule for multiplying two complex numbers (§ 124) gives at once for the *square of a complex number* $z = r(\cos\phi + i\sin\phi)$:

$$z^2 = [r(\cos\phi + i\sin\phi)]^2 = r^2(\cos 2\phi + i\sin 2\phi).$$

Multiplying both members by $z = r(\cos\phi + i\sin\phi)$ we find for the *cube:*

$$z^3 = [r(\cos\phi + i\sin\phi)]^3 = r^3(\cos 3\phi + i\sin 3\phi).$$

This suggests that we have generally for the nth *power* of z, n being any positive integer:

$$z^n = [r(\cos\phi + i\sin\phi)]^n = r^n(\cos n\phi + i\sin n\phi).$$

This is known as *de Moivre's formula.*

To complete the formal proof we use mathematical induction (§ 62). Assuming the formula to hold for some particular value of n, it is at once shown to hold for $n + 1$, by multiplying both members by

$$z = r(\cos\phi + i\sin\phi)$$

which gives

$$z^{n+1} = [r(\cos\phi + i\sin\phi)]^{n+1} = r^{n+1}[\cos(n+1)\phi + i\sin(n+1)\phi].$$

As the formula holds for $n = 2$, it holds for $n = 3$, and hence for $n = 4$, etc., *i.e.* for any positive integer.

127. Generalization of De Moivre's Theorem. De Moivre's formula can be shown to hold for any real exponent n. That it holds for a *negative* integer is seen as follows:

If in the formula for the quotient $z = z_1/z_2$ (§ 125) we put $r_1 = 1$, $\phi_1 = 0$, we find

$$\frac{1}{z_2} = \frac{1}{r_2}(\cos\phi_2 - i\sin\phi_2),$$

or dropping the subscript 2:

$$\frac{1}{z} = \frac{1}{r}(\cos\phi - i\sin\phi).$$

If we raise this complex number to the nth power (n being a positive integer), which can be done by § 126, we find

$$\left(\frac{1}{z}\right)^n = z^{-n} = \frac{1}{r^n}(\cos n\phi - i\sin n\phi),$$

which proves de Moivre's formula for a negative integral exponent.

If in de Moivre's formula (§ 126) we put

$$n\phi = \theta,\ r^n = \rho,\ \text{and hence}\ \phi = \frac{\theta}{n},\ r = \sqrt[n]{\rho},$$

where $\sqrt[n]{\rho}$ is the positive nth root of the real number ρ, we obtain

$$\left[\sqrt[n]{\rho}\left(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right)\right]^n = \rho(\cos\theta + i\sin\theta),$$

i.e.

$$[\rho(\cos\theta + i\sin\theta)]^{\frac{1}{n}} = \sqrt[n]{\rho}\left(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right).$$

This shows that de Moivre's formula holds when the exponent is of the form $1/n$. The extension to the case when the exponent is any rational fraction is then obvious.

128. Imaginary Roots. The last formula gives a means of finding *an nth root of any real or complex number.* To find *all* the roots of a complex number $z = \rho(\cos\theta + i\sin\theta)$ we must observe that as

$$\cos\theta = \cos(\theta + 2\pi m),\ \sin\theta = \sin(\theta + 2\pi m),$$

where m is any integer, the number z can be written in the form

$$z = \rho[\cos(\theta + 2\pi m) + i\sin(\theta + 2\pi m)],$$

so that by § 127 its roots are given by

$$\sqrt[n]{\rho}\left(\cos\frac{\theta+2\pi m}{n}+i\sin\frac{\theta+2\pi m}{n}\right).$$

If in this expression we give to m successively all integral values, it takes just n different values, viz. those for $m=0, 1, 2, \cdots, n-1$; therefore *any complex number* $z=\rho(\cos\theta+i\sin\theta)$ *has n roots*, viz.:

$$\sqrt[n]{\rho}\left(\cos\frac{\theta}{n}+i\sin\frac{\theta}{n}\right),\ \sqrt[n]{\rho}\left(\cos\frac{\theta+2\pi}{n}+i\sin\frac{\theta+2\pi}{n}\right),\ \cdots$$

$$\cdots\ \sqrt[n]{\rho}\left(\cos\frac{\theta+(n-1)2\pi}{n}+i\sin\frac{\theta+(n-1)2\pi}{n}\right).$$

These n roots all have the same modulus $\sqrt[n]{\rho}$, while the amplitudes differ by $2\pi/n$. Hence the points representing these n roots lie on a circle of radius $\sqrt[n]{\rho}$ about the origin and divide this circle into n equal parts.

For example, the three cube roots of $8i$ are found as follows. In polar form

$$0+8i=8(\cos\tfrac{1}{2}\pi+i\sin\tfrac{1}{2}\pi);$$

by de Moivre's formula (§ 127) we have

$$[8(\cos\tfrac{1}{2}\pi+i\sin\tfrac{1}{2}\pi)]^{\frac{1}{3}}$$

$$=2\left[\cos\frac{\frac{1}{2}\pi+2\pi m}{3}+i\sin\frac{\frac{1}{2}\pi+2\pi m}{3}\right],$$

$$=2[\cos(\tfrac{1}{6}\pi+\tfrac{2}{3}\pi m)+i\sin(\tfrac{1}{6}\pi+\tfrac{2}{3}\pi m)];$$

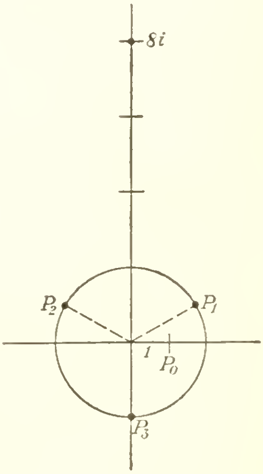


FIG. 51

$m=0$ gives the root:

$$2\left(\cos\frac{1}{6}\pi+i\sin\frac{1}{6}\pi\right)=2\left(\frac{\sqrt{3}}{2}+i\,\frac{1}{2}\right)=\sqrt{3}+i;$$

$m=1$ gives the root: $2(\cos\frac{5}{6}\pi+i\sin\frac{5}{6}\pi)=2(-\frac{1}{2}\sqrt{3}+i\,\frac{1}{2})=-\sqrt{3}+i$;
$m=2$ gives the root: $2(\cos\frac{3}{2}\pi+i\sin\frac{3}{2}\pi)=2(0+i(-1))=-2i$.

If we put $m=3$, we get the first root again, $m=4$ gives the second root, and so on. Thus there are three distinct cube roots of $8i$, viz. $\sqrt{3}+i$, $-\sqrt{3}+1$, $-2i$. These roots are represented by the points P_1, P_2, P_3, respectively (Fig. 51).

129. Square Roots. The particular problem of finding the *square root of a complex number* $a+bi$ can also be solved by observing that the problem requires us to find a complex number $x+yi$ such that

$$a+bi=(x+yi)^2.$$

Expanding the square and equating real and imaginary parts, we find for the determination of x and y the two equations

$$x^2-y^2=a, \qquad 2\,xy=b.$$

Eliminating y between these two equations, we obtain

$$x^2-\frac{b^2}{4\,x^2}=a\,; \text{ that is, } x^4-ax^2-\tfrac{1}{4}b^2=0\,;$$

whence $\qquad x_1^2=\frac{1}{2}(a+\sqrt{a^2+b^2}), \qquad x_2^2=\frac{1}{2}(a-\sqrt{a^2+b^2}).$

Since x is to be a real number and hence x^2 must be positive, and as $a<\sqrt{a^2+b^2}$ (unless $b=0$, which would mean that the given number $a+bi$ is real), we must take x_1^2 and not x_2^2. Hence

$$x=\pm\sqrt{\tfrac{1}{2}(a+\sqrt{a^2+b^2})}.$$

These values of x are zero only when $b=0$ and $a<0$; for then $\sqrt{a^2}=-a$. In this particular case we find $y=\pm\sqrt{-a}$, and hence

$$\sqrt{a+bi}=\pm\sqrt{-a}\;i.$$

In the general case, when $b\neq 0$, we find from the equation $2\,xy=b$ for each of the two values of x one value of y.

EXERCISES

1. Show how to locate the square of a complex number by geometric construction. Locate the cube.

2. Show geometrically that $8\,i$ (Fig. 51) is the product of the numbers represented by the points P_1, P_2, P_3.

3. For $z_1=1+2\,i$, $z_2=2-i$ show that $z_1^2-z_2^2=(z_1+z_2)(z_1-z_2)$ and illustrate geometrically.

4. For the same numbers verify and illustrate geometrically that $(z_1-z_2)^2=z_1^2-2\,z_1z_2+z_2^2$.

5. Show how to locate the points that represent the square roots of a complex number.

6. Locate by geometric construction in two ways the points which represent $[r(\cos\phi+i\sin\phi)]^{\frac{3}{2}}$.

7. Put the following complex numbers in polar form, perform the indicated operations, and check by geometric construction:

(*a*) $(1+\sqrt{3}\,i)^2$. (*b*) $(-1+i)^3$. (*c*) $(-\sqrt{3}-i)^2$.
(*d*) $(\sqrt{3}+i)^5$. (*e*) $(-1)^3$. (*f*) $(-i)^4$.
(*g*) $\sqrt{1+\sqrt{3}\,i}$. (*h*) $\sqrt[3]{-1-\sqrt{3}\,i}$. (*i*) $\sqrt[4]{-2+2\sqrt{3}\,i}$.
(*j*) $\sqrt[5]{-3-3\,i}$. (*k*) $\sqrt[3]{-4+4\,i}$. (*l*) $\sqrt[5]{64\,i}$.
(*m*) $\sqrt{-16\,i}$. (*n*) $\sqrt[3]{8\,i}$. (*o*) $\sqrt{(-3\,i)^3}$.

8. Find the square roots of each of the following complex numbers by using the method of § 129:

(*a*) $7+24\,i$. (*b*) $4\,i$. (*c*) $-2(8+15\,i)$.
(*d*) -16. (*e*) $\frac{1}{36}(5-12\,i)$. (*f*) $4\,ab+2(a^2-b^2)\,i$.
(*g*) $-2[2\,ab+(a^2-b^2)i]$. (*h*) $-4\,a^2b^2+2(a^4-b^4)i$.

9. Find the three cube roots of unity and show that either complex root is the square of the other, *i.e.* if one complex root of unity is denoted by ω, the other is ω^2. The three cube roots of unity then are 1, ω, ω^2.

10. If 1, ω, ω^2 are the cube roots of unity (see Ex. 9) show that:

(*a*) $1=\omega^3=\omega^6=\omega^{3n}$, n being an integer.
(*b*) $1+\omega+\omega^2=0$.
(*c*) $(1+\omega^2)^4=\omega$.
(*d*) $(\omega p+\omega^2 q)(\omega^2 p+\omega q)(p+q)=p^3+q^3$.
(*e*) $(1-\omega+\omega^2)(1+\omega-\omega^2)=4$.
(*f*) $(1-\omega+\omega^2)(1-\omega^2+\omega^4)(1-\omega^4+\omega^8)=-8\,\omega$.

11. Prove de Moivre's formula for n any rational fraction, *i.e.* show that, if p, q, m, are integers,

$$[r(\cos\phi+i\sin\phi)]^{\frac{p}{q}}=r^{\frac{p}{q}}\left[\cos\frac{p\phi+2\,\pi m}{q}+i\sin\frac{p\phi+2\,\pi m}{q}\right].$$

12. Show by geometric construction that the sum of the three cube roots of any number is equal to zero; that the sum of the four fourth roots is zero.

13. Solve the following equations and locate the points which represent the roots:

(*a*) $x^2-1=0$. (*b*) $x^3+1=0$. (*c*) $x^4-1=0$. (*d*) $x^5-1=0$.
(*e*) $x^6-1=0$. (*f*) $x^3-27=0$. (*g*) $x^2+1=0$. (*h*) $x^4+16=0$.
(*i*) $x^5+32=0$. (*j*) $x^2+a^2=0$. (*k*) $x^3+a^3=0$. (*l*) $x^8-1=0$.

CHAPTER VIII

POLYNOMIALS. NUMERICAL EQUATIONS

PART I. QUADRATIC FUNCTION — PARABOLA

130. Linear Function. As mentioned in § 28, an expression of the form $mx+b$, where m and b are given real numbers ($m \neq 0$) while x may take any real value, is called a *linear function* of x. We have seen that this function is represented graphically by the ordinates of the *straight line*

$$y = mx + b;$$

b is the value of y for $x = 0$, and m is the *slope* of the line, *i.e.* the *rate of change* of the function y with respect to x.

131. Quadratic Function. Parabola. An expression of the form ax^2+bx+c in which $a \neq 0$ is called a ***quadratic function*** of x, and the curve

$$y = ax^2 + bx + c,$$

whose ordinates represent the function, is called a ***parabola.***

If the *coefficients* a, b, c are given numerically, any number of points of this curve can be located by arbitrarily assigning to the abscissa x any series of values and computing from the equation the corresponding values of the ordinates. This process is known as *plotting the curve by points*; it is somewhat laborious; but a study of the nature of the quadratic function will show that the determination of a few points is sufficient to give a good idea of the curve.

132. The Form $y = ax^2$. Let us first take $b = 0$, $c = 0$; the resulting equation

(1) $$y = ax^2$$

represents a parabola which passes through the origin, since the values 0, 0 satisfy the equation. *This parabola is symmetric with respect to the axis Oy;* for, the values of y corresponding to any two equal and opposite values of x are equal. This line of symmetry is called the ***axis*** of the parabola; its intersection with the parabola is called the ***vertex***.

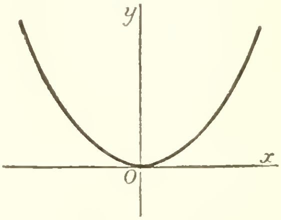


FIG. 52

We may distinguish two cases according as $a > 0$ or $a < 0$; if $a = 0$, the equation becomes $y = 0$, which represents the axis Ox.

(1) If $a > 0$, the curve lies above the axis Ox. For, no matter what positive or negative value is assigned to x, y is positive. Furthermore, as x is allowed to increase in absolute value, y also increases indefinitely. Hence the parabola lies in the first and second quadrants with its vertex at the origin and *opens upward*, i.e. *is concave upward* (Fig. 52).

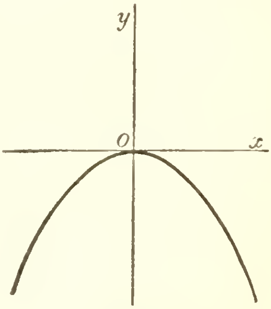


FIG. 53

(2) If $a < 0$, we conclude, similarly, that the parabola lies below the axis Ox, in the third and fourth quadrants, with its vertex at the origin and *opens downward*, i.e. *is concave downward* (Fig. 53).

Draw the following parabolas:

$$y = x^2,\ y = 3\,x^2,\ y = -\tfrac{1}{2}\,x^2,\ y = \tfrac{1}{4}\,x^2.$$

133. The General Equation. The curve represented by the more general equation

(2) $$y = ax^2 + bx + c$$

differs from the parabola $y = ax^2$ only in position. To see this we use the process of *completing the square in x*; *i.e.* we write the equation in the equivalent form

$$y = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c;$$

i.e.

$$y - \left(-\frac{b^2}{4a} + c\right) = a\left(x + \frac{b}{2a}\right)^2.$$

If we put

$$h = -\frac{b}{2a},\ k = -\frac{b^2}{4a} + c,$$

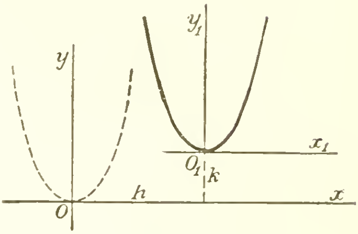


FIG. 54

the equation becomes

$$y - k = a(x - h)^2,$$

and it is clear (§ 13) that, with reference to parallel axes O_1x_1, O_1y_1 through the point O_1 (h, k) the equation of the curve is $y_1 = ax_1^2$ (Fig. 54). The parabola (2) has therefore the same shape as the parabola $y = ax^2$; but its vertex lies at the point (h, k), and its axis is the line $x = h$. The curve opens upward or downward according as $a > 0$ or $a < 0$.

134. Nature of the Curve. To sketch the parabola (2) roughly, it is often sufficient to find the vertex (by completing the square in x, as in § 133), and the intersections with the axes. The intercept on the axis Oy is obviously equal to c. The intercepts on the axis Ox are found by solving the quadratic equation

$$ax^2 + bx + c = 0.$$

We have thus an interesting interpretation of the roots of any quadratic equation: the roots of $ax^2 + bx + c = 0$ are the abscissas of the points at which the parabola (2) intersects the axis Ox. The ordinate of the vertex of the parabola is evidently the least or greatest value of the function $y = ax^2 + bx + c$ according as a is greater or less than zero.

EXERCISES

1. With respect to the same coordinate axes draw the curves $y = ax^2$ for $a = 2, \frac{3}{2}, 1, \frac{1}{2}, 0, -\frac{1}{2}, -1, -\frac{3}{2}, -2$. What happens to the parabola $y = ax^2$ as a changes?

2. Determine in each of the following examples the value of a so that the parabola $y = ax^2$ will pass through the given point:

(*a*) $(2, 3)$. (*b*) $(-4, 1)$. (*c*) $(-2, -2)$. (*d*) $(3, -4)$.

3. A body thrown vertically upward in a vacuum with a velocity of v feet per second will just reach a height of h feet such that $h = \frac{1}{64}v^2$. Draw the curve whose ordinates represent the height as a function of the initial velocity.

(*a*) With what velocity must a ball be thrown vertically upward to rise to a height of 100 ft.?

(*b*) How high will a bullet rise if shot vertically upward with an initial velocity of 800 ft. per sec., the resistance of the air being neglected?

4. The period of a pendulum of length l (*i.e.* the time of a small back and forth swing) is $T = 2\pi\sqrt{l/g}$. Take $g = 32$ ft./sec. and draw the curve whose ordinates represent the length l of the pendulum as a function of the period T.

(*a*) How long is a pendulum that beats seconds (*i.e.* of period 2 sec.)?

(*b*) How long is a pendulum that makes one swing in two seconds?

(*c*) Find the period of a pendulum of length one yard.

5. Draw the following parabolas and find their vertices and axes:

(*a*) $y = \frac{1}{4}x^2 - x + 2$. (*b*) $y = -\frac{1}{4}x^2 + x$. (*c*) $y = 5x^2 + 15x + 3$.
(*d*) $y = 2 - x - x^2$. (*e*) $y = x^2 - 9$. (*f*) $y = -9 - x^2$.
(*g*) $y = 3x^2 - 6x + 5$. (*h*) $y = \frac{1}{2}x^2 + 2x - 6$. (*i*) $x^2 - 2x - y = 0$.

6. What is the value of b if the parabola $y = x^2 + bx - 6$ passes through the point $(1, 5)$? of c if the parabola $y = x^2 - 6x + c$ passes through the same point?

7. Suppose the parabola $y = ax^2$ drawn; how would you draw $y = a(x+2)^2$? $y = a(x-7)^2$? $y = ax^2 + 2$? $y = ax^2 - 7$? $y = ax^2 + 2x + 3$?

8. What happens to the parabola $y = ax^2 + bx + c$ as c changes? For example, take the parabola $y = x^2 - x + c$, where $c = -3, -2, -1, 0, 1, 2, 3$.

9. What happens to the parabola $y = ax^2 + bx + c$ as a changes? For example, take $y = ax^2 - x - 6$, where $a = 2, 1, \frac{1}{2}, 0, -\frac{1}{2}, -1, -2$.

10. (a) If the parabola $y = ax^2 + bx$ is to pass through the points $(1, 4)$, $(-2, 1)$ what must be the values of a and b? (b) Determine the parabola $y = ax^2 + bx + c$ so as to pass through the points $(1, \frac{1}{2})$, $(3, 2)$, $(4, \frac{3}{2})$; sketch.

11. The path of a projectile in a vacuum is a parabola with vertical axis, opening downward. With the starting point of the projectile as origin and the axis Ox horizontal, the equation of the path must be of the form $y = ax^2 + bx$. If the projectile is observed to pass through the points $(30, 20)$ and $(50, 30)$, what is the equation of the path? What is the highest point reached? Where will the projectile reach the ground?

12. Find the equations of the parabolas determined by the following conditions:

(a) the axis coincides with Oy, the vertex is at the origin, and the curve passes through the point $(-2, -3)$;

(b) the axis is the line $x = 3$, the vertex is at $(3, -2)$, and the curve passes through the origin;

(c) the axis is the line $x = -4$, the vertex is $(-4, 6)$, and the curve passes through the point $(1, 2)$.

13. Sketch the following parabolas and lines and find the coordinates of their points of intersection:

(a) $y = 6x^2$, $y = 7x + 3$.
(b) $y = 2x^2 - 3x$, $y = x + 6$.
(c) $y = 2 - 3x^2$, $y = 2x + 3$.
(d) $y = 3 + x - x^2$, $x + y - 4 = 0$.

14. Sketch the following curves and find their intersections:

(a) $x^2 + y^2 = 25$, $y = \frac{4}{9}x^2$.
(b) $x^2 + y^2 - 6y = 0$, $y = \frac{1}{3}x^2 - 2x + 6$.

15. The ordinate of every point of the line $y = \frac{2}{3}x + 4$ is the sum of the corresponding ordinates of the lines $y = \frac{2}{3}x$ and $y = 4$. Draw the last two lines and from them construct the first line.

16. The ordinate of every point of the parabola $y = \frac{1}{2}x^2 + \frac{1}{2}x - 1$ is the sum of the corresponding ordinates of the parabola $y = \frac{1}{2}x^2$ and the line $y = \frac{1}{2}x - 1$. From this fact draw the former parabola.

17. The ordinate of every point of the parabola $y = \frac{1}{3}x^2 - x + 3$ is the difference of the corresponding ordinates of the parabola $y = \frac{1}{3}x^2$ and the line $y = x - 3$. In this way sketch the former parabola.

18. Suppose the parabola $y = ax^2 + bx + c$ drawn, how would you sketch the following curves? Are these curves also parabolas?

(*a*) $y = a(x + h)^2 + b(x + h) + c, h > 0.$
(*b*) $y = a(x - 2)^2 + b(x - 2) + c.$
(*c*) $y = a(2x)^2 + b(2x) + c.$
(*d*) $y = a(\frac{1}{3}x)^2 + b(\frac{1}{3}x) + c.$

19. Find the values of x for which the following relations are true:

(*a*) $x^2 - x - 12 < 0.$
(*b*) $12 - x - x^2 > 0.$
(*c*) $3x^2 + 5x - 2 \gtreqless 0.$
(*d*) $5 + 13x - 6x^2 \gtreqless 0.$
(*e*) $x^2 - 5 > 3x + 5.$
(*f*) $x^2 - 5 < 3x + 5.$

20. Show that the equation of the parabola $y = ax^2 + bx + c$ that passes through the points (x_1, y_1), (x_2, y_2), (x_3, y_3) may be written in the form

$$\begin{vmatrix} y & x^2 & x & 1 \\ y_1 & x_1^2 & x_1 & 1 \\ y_2 & x_2^2 & x_2 & 1 \\ y_3 & x_3^2 & x_3 & 1 \end{vmatrix} = 0.$$

(*a*) Show that if the minor of x^2 vanishes, the three given points lie on a line.

(*b*) What conclusion do you draw if the minor of y vanishes?

(*c*) To what does this determinant reduce if the origin is one of the given points?

135. Symmetry. Two points P_1, P_2 are said to be situated *symmetrically* with respect to a line l, if l is the perpendicular bisector of P_1P_2; this is also expressed by saying that either point is the *reflection* of the other in the line l.

Any two plane figures are said to be symmetric with respect to a line l in their plane if either figure is formed of the reflections in l of all the points of the other figure. Each figure is then the reflection of the other in the line l. Two such figures are evidently brought to coincidence by turning either figure about the line l through two right angles. Thus, the lines $y = 2x + 3$ and $y = -2x - 3$ are symmetric with respect to the axis Ox.

A line l is called an *axis of symmetry* (or simply an *axis*) of a figure if the portion of the figure on one side of l is the reflection in l of the portion on the other side. Thus, any diameter of a circle is an axis of symmetry of the circle. What are the axes of symmetry of a square? of a rectangle? of a parallelogram?

In analytic geometry, symmetry with respect to the axes of coordinates, and to the lines $y = \pm x$, is of particular importance.

It is readily seen that if a figure is symmetric with respect to *both* axes of coordinates, it is *symmetric with respect to the origin, i.e.* to every point P_1 of the figure there exists another point P_2 of the figure such that the origin bisects P_1P_2. A point of symmetry of a figure is also called ***center*** of the figure.

EXERCISES

1. Give the coordinates of the reflection of the point (a, b) in the axis Ox; in the axis Oy; in the line $y = x$; in the line $y = 2x$; in the line $y = -x$.

2. Show that when x is replaced by $-x$ in the equation of a given curve, we obtain the equation of the reflection of the given curve in the y-axis.

3. Show that when x and y are replaced by y and x, respectively, in the equation of a given curve, we obtain the equation of the reflection of the given curve in the line $y = x$.

4. Sketch the lines $y = -2x + 5$ and $x = -2y + 5$ and find their point of intersection.

5. Sketch the parabolas $y = x^2$ and $x = y^2$ and find their points of intersection.

6. Find the equation of the reflection of the line $2x - 3y + 4 = 0$ in the line $y = x$; in the axis Ox; in the axis Oy; in the line $y = -x$.

7. What is the reflection of the line $x = a$ in the line $y = x$? in the axes?

8. Find and sketch the circle which is the reflection of the circle $x^2 + y^2 - 3x - 2 = 0$ in the line $y = x$, and find the points in which the two circles intersect.

9. Find the circle which is the reflection of the circle $x^2 + y^2 - 4x + 3 = 0$ in the line $y = x$; in the coordinate axes. Sketch all of these circles.

10. What is the general equation of a circle which is its own reflection in the line $y = x$? in the axis Ox? in the axis Oy? What circle is its own reflection in all three of these lines?

11. What is the equation of the reflection of the parabola $y = -x^2 + 4$ in the line $y = x$? in the line $y = -x$? Are these reflections parabolas?

12. What is the reflection of the parabola $y = 3x^2 - 5x + 6$ in the axis Ox? in the axis Oy? Are these reflections parabolas?

13. By drawing accurately the parabolas $y + x^2 = 7$, $x + y^2 = 11$, find approximately the coordinates of their points of intersection.

14. If the Cartesian equation of a curve is not changed when x is replaced by $-x$, the curve is symmetric with respect to Oy; if it is not changed when y is replaced by $-y$, the curve is symmetric with respect to Ox; if it is not changed when x and y are replaced by $-x$ and $-y$, respectively, the curve is symmetric with respect to the origin; if it is not changed when x and y are interchanged, the curve is symmetric with respect to $y = x$.

136. Slope of Secant. Let $P(x, y)$ be any point of the parabola

(1) $$y = ax^2.$$

If $P_1(x_1, y_1)$ be any other point of this parabola so that

(2) $$y_1 = ax_1^2,$$

the line PP_1 (Fig. 55) is called a *secant*.

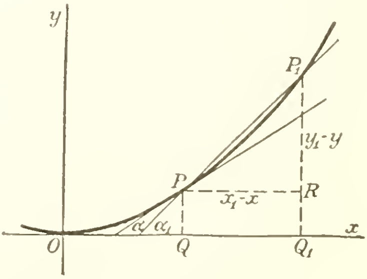


FIG. 55

For the *slope* tan α_1 *of this secant* we have from Fig. 55:

(3) $$\tan\alpha_1 = \frac{RP_1}{QQ_1} = \frac{y_1 - y}{x_1 - x} = \frac{\Delta y}{\Delta x},$$

or, substituting for y and y_1 their values:

(4) $$\tan\alpha_1 = \frac{a(x_1^2 - x^2)}{x_1 - x} = a(x + x_1).$$

137. Slope of Tangent. Keeping the point P (Fig. 55) fixed, let the point P_1 move along the parabola toward P; the limiting position which the secant PP_1 assumes at the instant when P_1 passes through P is called the ***tangent*** to the parabola at the point P.

Let us determine the *slope* $\tan\alpha$ *of this tangent.* As the secant turns about P approaching the tangent, the point Q_1 approaches the point Q, and in the limit $OQ_1 = x_1$ becomes $OQ = x$. The last formula of § 136 gives therefore $\tan\alpha$ if we make $x_1 = x$:

$$\tan\alpha = 2\,ax.$$

The slope of the tangent at P which indicates the "steepness" of the curve at P is also called the *slope of the parabola* at P. Thus the slope of the parabola $y = ax^2$ at any point whose abscissa is x is $= 2\,ax$; notice that it varies from point to point, being a function of x, while the slope of a straight line is constant all along the line.

The knowledge of the slope of a curve is of great assistance in sketching the curve because it enables us, after locating a number of points, to draw the tangent at each point. Thus, for the parabola $y = \frac{2}{3}x^2$ we find $\tan\alpha = \frac{4}{3}x$; locate the points for which $x = 0, 1, 2, -1, -2$, and draw the tangents at these points; then sketch in the curve.

138. Derivative. If we think of the ordinate of the parabola $y = ax^2$ as representing the function ax^2, the slope of the parabola represents the rate at which the function varies with x and is called the ***derivative*** of the function ax^2. We shall denote the derivative of y by y'. In § 137 we have proved that the derivative of the function

$$y = ax^2,$$

is

$$y' = 2\,ax.$$

The process of finding the derivative of a function, which is called *differentiation*, consists, according to §§ 136–137, in the following steps: Starting with the value $y=ax^2$ of the function for some particular value of x (say, at the point P, Fig. 55), we give to x an *increment* $x_1-x=\Delta x$ (compare § 9) and calculate the value of the corresponding increment $y_1-y=\Delta y$ of the function. Then *the derivative y' of the function y is the limit that $\Delta y / \Delta x$ approaches as Δx approaches zero.* In the case of the function $y=ax^2$ we have

$$\Delta y=y_1-y=a(x_1^2-x^2)=a[(x+\Delta x)^2-x^2]=a[2\,x\Delta x+(\Delta x)^2];$$

hence $$\frac{\Delta y}{\Delta x}=a(2\,x+\Delta x).$$

The limit of the right-hand member as Δx approaches zero gives the derivative:

$$y'=2\,ax.$$

Thus, the area y of a circle in terms of its radius x is

$$y=\pi x^2.$$

Hence the *derivative* y', that is the slope of the tangent to the curve that represents the equation $y=\pi x^2$, is

$$y'=2\,\pi x.$$

This represents (§ 137) the *rate of increase* of the area y with respect to x. Since $2\,\pi x$ is the length of the circumference, we see that the rate of increase of the area y with respect to the radius x is equal to the circumference of the circle.

139. Derivative of General Quadratic Function. By this process we can at once find the derivative of the general quadratic function $y=ax^2+bx+c$ (§ 131), and hence the slope of the parabola represented by this equation. We have here

$$\begin{aligned}\Delta y&=a(x+\Delta x)^2+b(x+\Delta x)+c-(ax^2+bx+c)\\&=2\,ax\Delta x+a(\Delta x)^2+b\Delta x;\end{aligned}$$

hence $$\frac{\Delta y}{\Delta x}=2\,ax+b+a\Delta x.$$

The limit, as Δx becomes zero, is $2ax + b$; hence *the derivative of the quadratic function* $y = ax^2 + bx + c$ *is*

$$y' = 2ax + b.$$

140. Maximum or Minimum Value. It follows both from the definition of the derivative as the limit of $\Delta y/\Delta x$ and from its geometrical interpretation as the slope, $\tan\alpha$, of the curve that *if, for any value of* x, *the derivative is positive, the function,* *i.e.* the ordinate of the curve, *is (algebraically) increasing;* if the derivative is negative, the function is decreasing.

At a point where the derivative is zero the tangent to the curve is parallel to the axis Ox. The abscissas of the points at which the tangent is parallel to Ox can therefore be found by equating the derivative to zero. In this way we find that the abscissa of the vertex of the parabola $y = ax^2 + bx + c$ is

$$x = -\frac{b}{2a},$$

which agrees with § 133.

We know (§ 133) that the parabola $y = ax^2 + bx + c$ opens upward or downward according as a is > 0 or < 0. Hence the ordinate of the vertex is a *minimum ordinate, i.e.* algebraically less than the immediately preceding and following ordinates, if $a > 0$; it is a *maximum ordinate, i.e.* algebraically greater than the immediately preceding and following ordinates, if $a < 0$.

We have thus a simple method for determining the maximum or minimum of a quadratic function $ax^2 + bx + c$; the value of x for which the function becomes greatest or least is found by equating the derivative to zero; the quadratic function is a maximum or a minimum for this value of x according as $a < 0$ or > 0.

Thus, to determine the greatest rectangular area that can be inclosed by a boundary (*e.g.* a fence) of given length $2k$, let one side of the rectangle be called x: then the other side is $k - x$. Hence the area A of the rectangle is

$$A = x(k - x) = kx - x^2.$$

Consequently the derivative of A is $k - 2x$. If we set this equal to zero, we have $2x = k$, whence $x = k/2$. It follows that $k - x = k/2$; hence the rectangle of greatest area is a square

EXERCISES

1. Locate the points of the parabola $y = x^2 - 4x + \frac{5}{2}$ whose abscissas are -1, 0, 1, 2, 3, 4, draw the tangents at these points, and then sketch in the curve.

2. Sketch the parabolas $4y = -x^2 + 4x$ and $y = x^2 - 3$ by locating the vertex and the intersections with Ox and drawing the tangents at these points.

3. Is the function $y = 5(x^2 - 4x + 3)$ increasing or decreasing as x increases from $x = \frac{1}{2}$? from $x = \frac{5}{2}$?

4. Find the least or greatest value of the quadratic functions:

(*a*) $2x^2 - 3x + 6$. (*b*) $8 - 6x - x^2$. (*c*) $x^2 - 5x - 5$.
(*d*) $2 - 2x - x^2$. (*e*) $4 + x - \frac{1}{3}x^2$. (*f*) $5x^2 - 20x + 1$.

5. Find the derivative of the linear function $y = mx + b$.

6. The curve of a railroad track is represented by the equation $y = \frac{1}{3}x^2$, the axes Ox, Oy pointing east and north, respectively; in what direction is the train going at the points whose abscissas are 0, 1, 2, $-\frac{1}{2}$?

7. A projectile describes the parabola $y = \frac{4}{7}x - 3x^2$, the unit being the mile. What is the angle of elevation of the gun? What is the greatest height? Where does the projectile strike the ground?

8. A rectangular area is to be inclosed on three sides, the fourth side being bounded by a straight river. If the length of the fence is a constant k, what is the maximum area of the rectangle?

9. Let e denote the error made in measuring the side of a square of 100 sq. ft. area, and E the corresponding error in the computed area. Draw the curve representing E as a function of e.

10. A rectangle surmounted by a semicircle has a total perimeter of 100 inches. Draw the curve representing the total area as a function of the radius of the semicircle. For what radius is the area greatest?

PART II. CUBIC FUNCTION

141. The Cubic Function. A function of the form $a_0x^3 + a_1x^2 + a_2x + a_3$ is called a *cubic function* of x. The curve represented by the equation

$$y = a_0x^3 + a_1x^2 + a_2x + a_3$$

can be sketched by plotting it by points (§ 131).

For example, to draw the curve represented by the equation

$$y = x^3 - 2x^2 - 5x + 6,$$

we select a number of values of x and compute the corresponding values of y:

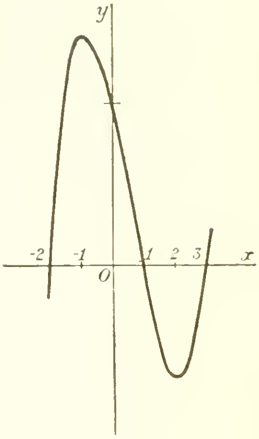


FIG. 56

$x =$	-3	-2	-1	0	1	2	3	4	$\cdots$
$y =$	-24	0	8	6	0	-4	0	18	$\cdots$

These points can then be plotted and connected by a smooth curve which will approximately represent the curve corresponding to the given equation (Fig. 56).

142. Derivative. The sketching of such a cubic curve is again greatly facilitated by finding the derivative of the cubic function; the determination of a few points, with their tangents, will suffice to give a good general idea of the curve.

To find the ***derivative*** of the function $y = a_0x^3 + a_1x^2 + a_2x + a_3$ the process of § 138 should be followed. The student may carry this out himself; he will find the quadratic function

$$y' = 3a_0x^2 + 2a_1x + a_2.$$

143. Maximum or Minimum Values. The abscissas of those points of the curve at which the tangent is parallel to the axis Ox are again found by equating the derivative to zero; they are therefore the roots of the quadratic equation

$$3a_0x^2 + 2a_1x + a_2 = 0.$$

If at such a point the derivative passes from positive to negative values, the curve is *concave downward*, and the ordinate is a *maximum*; if the derivative passes from negative to positive values, the curve is *concave upward*, and the ordinate is a *minimum*.

144. Second Derivative. The derivative of a function of x is in general again a function of x. Thus for the cubic function $y = a_0x^3 + a_1x^2 + a_2x + a_3$ the derivative is the quadratic function

$$y' = 3a_0x^2 + 2a_1x + a_2.$$

The *derivative of the first derivative* is called the ***second derivative*** of the original function; denoting it by y'', we find (§ 139)

$$y'' = 6a_0x + 2a_1.$$

As a positive derivative indicates an increasing function, while a negative derivative indicates a decreasing function (§ 140), it follows that if at any point of the curve the second derivative is positive, the first derivative, *i.e.* the slope of the curve, increases; geometrically this evidently means that the curve there is *concave upward*. Similarly, if the second derivative is negative, the curve is *concave downward*. We have thus a simple means of telling whether at any particular point the curve is concave upward or downward.

It follows that at any point where the first derivative vanishes, the ordinate is a *minimum* if the second derivative is positive; it is a *maximum* if the second derivative is negative.

145. Points of Inflexion. A point at which the curve changes from being concave downward to being concave upward, or *vice versa*, is called a ***point of inflexion***. At such a point the second derivative vanishes.

Our cubic curve obviously has but one point of inflection, viz. the point whose abscissa is $x = -a_1/(3a_0)$.

EXERCISES

1. Find the first and second derivatives of y when:

(*a*) $y = 6x^3 - 7x^2 - x + 2$. (*b*) $y = 20 + 4x - 5x^2 - x^3$.
(*c*) $10y = x^3 - 5x^2 + 3x + 9$. (*d*) $y = (x-1)(x-2)(x-3)$.
(*e*) $y = x^2(x+3)$. (*f*) $7y = 3x - 2x(x^2 - 1)$.

2. Sketch the curve $y = (x-2)(x+1)(x+3)$, observing the sign of y between the intersections with Ox, and determining the minimum, maximum, and point of inflection.

3. In the curve $y = a_0x^3 + a_1x^2 + a_2x + a_3$, what is the meaning of a_3?

4. Sketch the curves:

(*a*) $5y = (x-1)(x+4)^2$. (*b*) $y = (x-3)^3$.
(*c*) $6y = 6 + x + x^2 - x^3$. (*d*) $y = x^3 - 4x$.
(*e*) $8y = 5x^2 - x^3$. (*f*) $y = x^3 - 3x^2 + 4x - 5$.

5. Draw the curves $y = x$, $y = x^2$, $y = x^3$, with their tangents at the points whose abscissas are 1 and -1.

6. Find the equation of the tangent to the curve $14y = 5x^3 - 2x^2 + x - 20$ at the point whose abscissa is 2.

7. At what points of the curve $y = x^3 - 5x^2 + 3$ are the tangents parallel to the line $y = -3x + 5$?

8. Are the following curves concave upward or downward at the indicated points? Sketch each of them.

(*a*) $y = 4x^3 - 6x$, at $x = 3$. (*b*) $3y = 5x - 3x^3$, at $x = -2$.
(*c*) $y = x^3 - 2x^2 + 5$, at $x = \frac{1}{3}$. (*d*) $2y = x^3 - 3x^2$, at $x = 1$.
(*e*) $y = 1 - x - x^3$, at $x = 0$. (*f*) $10y = x^3 + x^2 - 15x + 6$, at $x = -\frac{1}{3}$.

9. Show that the parabola $y = ax^2 + bx + c$ is concave upward or concave downward for all values of x according as a is positive or negative.

10. The angle between two curves at a point of intersection is the angle between their tangents. Find the angles between the curves $y = x^2$ and $y = x^3$ at their points of intersection.

11. Find the angle at which the parabola $y = 2x^2 - 3x - 5$ intersects the curve $y = x^3 + 3x - 17$ at the point $(2, -3)$.

12. The ordinate of every point of the curve $y = x^3 + 2x^2$ is the sum of the ordinates of the curves $y = x^3$ and $y = 2x^2$. From the latter two curves construct the former.

13. From the curve $y = x^3$ construct the following curves:

(*a*) $y = 4x^3$. (*b*) $y = \left(\frac{x}{2}\right)^3$. (*c*) $y = x^3 - 2$. (*d*) $y = 2x^3 + 4$.

14. Draw the curve $2y = x^3 - 3x^2$ and its reflection in the line $y = x$. What is the equation of this reflected curve? What is the equation of the reflection in the axis Oy?

15. A piece of cardboard 18 inches square is used to make a box by cutting equal squares from the four corners and turning up the sides. Draw the curve whose ordinates represent the volume of the box as a function of the side of the square cut out. Find its maximum.

16. The strength of a rectangular beam cut from a log one foot in diameter is proportional to (*i.e.* a constant times) the width and the square of the depth. Find the dimensions of the strongest beam which can be cut from the log. Draw the curve whose ordinates represent the strength of the beam as a function of the width.

17. Show that the equation of a curve in the form $y = ax^3 + bx^2 + cx + d$ is in general determined by four points $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, $P_3(x_3, y_3)$, $P_4(x_4, y_4)$, and may be written in the form

$$\begin{vmatrix} y & x^3 & x^2 & x & 1 \\ y_1 & x_1^3 & x_1^2 & x_1 & 1 \\ y_2 & x_2^3 & x_2^2 & x_2 & 1 \\ y_3 & x_3^3 & x_3^2 & x_3 & 1 \\ y_4 & x_4^3 & x_4^2 & x_4 & 1 \end{vmatrix} = 0.$$

18. Find the equation of the curve in the form $y = ax^3 + bx^2 + cx + d$ which passes through the following points:

(*a*) $(0, 0)$, $(2, -1)$, $(-1, 4)$, $(3, 4)$;
(*b*) $(1, 1)$, $(3, -1)$, $(0, 5)$, $(-4, 1)$.

19. Show that every cubic curve of the form $y = a_0x^3 + a_1x^2 + a_2x + a_3$ is symmetric with respect to its point of inflection.

146. Cubic Equation. The real roots of the cubic equation

$$a_0x^3 + a_1x^2 + a_2x + a_3 = 0$$

are the abscissas of the points at which the cubic curve

$$y = a_0x^3 + a_1x^2 + a_2x + a_3$$

intersects the axis Ox. This geometric interpretation can be used to find the real roots of a *numerical* cubic equation approximately: calculating* the ordinates for a series of values of x (as in plotting the curve by points, § 141), or at least determining the *signs* of these ordinates, observe where the ordinate changes sign. At least one real root must lie between any two values of x for which the ordinates have opposite signs. The first approximation so obtained can then be improved by calculating ordinates for intermediate values of x.

Thus to find the roots of the cubic

$$x^3 + x^2 - 16x + 6 = 0$$

we find that

for $x =$	-5	-4	-3	-2	-1	0	1	2	3	4
y is	$-$	$+$	$+$	$+$	$+$	$+$	$-$	$-$	$-$	$+$

The roots lie therefore between -5 and -4, 0 and 1, 3 and 4. To find, *e.g.*, the root that lies between 0 and 1, we find that

for $x =$	0	0.1	0.2	0.3	0.4
y is	$+$	$+$	$+$	$+$	$-$

The root lies therefore between 0.3 and 0.4, and as the corresponding values of y are 1.317 and -0.176, the root is somewhat less than 0.4. As

for $x =$	0.40	0.39	0.38
$y =$	-0.176	-0.029	$+0.119$

a more accurate value of the root is 0.39.

This process can be carried as far as we please. But it is very laborious. We shall see in a later section (§ 170) how it can be systematized.

EXERCISES

1. Find to three significant figures the real roots of the equations:

(*a*) $x^3 - 4x^2 + 5 = 0.$ (*b*) $x^3 + x^2 - x - \frac{1}{2} = 0.$

(*c*) $x^3 - 3x + 1\frac{1}{2} = 0.$ (*d*) $x(x-1)(x-2) = 4.$

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* For abridged numerical multiplication and division see the note on p. 256.

PART III. THE GENERAL POLYNOMIAL

147. Polynomials. The methods used in studying the quadratic and cubic functions and the curves represented by them can readily be extended to the general case of the ***polynomial***, or *rational integral function*, of the nth degree,

$$y = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n,$$

where the coefficients $a_0, a_1, \cdots a_n$ may be any real numbers, while the exponent n, which is called the *degree* of the polynomial, is a positive integer.

We shall often denote such a polynomial by the letter y or by the symbol $f(x)$ (read: function of x, or f of x); its value for any particular value of x, say $x = x_1$ or $x = h$, is then denoted by $f(x_1)$ or $f(h)$, respectively. Thus, for $x = 0$ we have $f(0) = a_n$.

148. Calculation of Values of a Polynomial. In plotting the curve $y = f(x)$ by points (§§ 131, 141) we have to calculate a number of ordinates. Unless $f(x)$ is a very simple polynomial this is a rather laborious process. To shorten it observe that the value $f(x_1)$ of the polynomial

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$$

for $x = x_1$ can be written in the form

$$f(x_1) = (\cdots(((a_0x_1 + a_1)x_1 + a_2)x_1 + a_3)x_1 + \cdots + a_{n-1})x_1 + a_n.$$

To calculate this expression begin by finding $a_0x_1 + a_1$; multiply by x_1 and add a_2; multiply the result by x_1 and add a_3; etc. This is best carried out in the following form:

$$\begin{array}{llll}
a_0 & a_1 & a_2 & \cdots a_n \\
 & a_0x_1 & (a_0x_1 + a_1)x_1 & \\
 & a_0x_1 + a_1 & (a_0x_1 + a_1)x_1 + a_2 & \cdots
\end{array}$$

For instance, if

$$\begin{aligned} f(x) &= 2x^3 - 3x^2 - 12x + 5 \\ &= ((2x - 3)x - 12)x + 5, \end{aligned}$$

to find $f(3)$ write the coefficients in a row and place $2 \times 3 = 6$ below the second coefficient; the sum is 3. Place $3 \times 3 = 9$ below the third coefficient: the sum is -3. Place $3 \times (-3) = -9$ below the last coefficient: the sum, -4, is $= f(3)$.

$$\begin{array}{rrrr} 2 & -3 & -12 & 5 \\ & 6 & 9 & -9 \\ \hline 2 & 3 & -3 & -4 \end{array}$$

This process is useful in calculating the values of y that correspond to various values of x, as we have to do in plotting a curve by points. It is also very convenient in solving an equation by the method of § 146.

EXERCISES

1. If $f(x) = 5x^3 - 13x + 2$, what is meant by $f(a)$? by $f(x + h)$? What is the value of $f(0)$? of $f(2)$? of $f(-3/5)$? of $f(-1)$?

2. Find the ordinates of the curve $y = x^4 - x^3 + 3x^2 - 12x + 3$ for $x = 3, -9, -\frac{1}{2}$.

3. Find the ordinates of $2y = x^4 + 3x^2 - 20x - 25$ for $x = 1, 2, 3, -1, -2$.

4. Suppose the curve $y = f(x)$ drawn; how would you sketch:

(a) $y = f(x - 2)$? (b) $y = f(x + 3)$? (c) $y = f(2x)$? (d) $y = f(-x)$?

(e) $y = f\left(\dfrac{x}{4}\right)$? (f) $y = f(x) + 5$? (g) $y = f(x) - 2x$?

5. Calculate to three places of decimals the real roots of the equations:

(a) $x^3 + x^2 = 100$; (b) $x^3 - 4 = 0$; (c) $x^3 - 7x + 7 = 0$.

149. Derivative of the Polynomial. We have seen in the preceding sections how greatly the sketching of a curve and the investigation of a function is facilitated by the use of the derivatives of the function. Thus, in particular, the ***first derivative*** y' is the ***rate of change*** of the function y with x, and hence determines the slope, or steepness, of the curve $y = f(x)$. We begin therefore the study of the polynomial by determining its derivative. The method is essentially the same as that used in §§ 138, 139 for finding the derivative of a quadratic function.

The ***first derivative*** y' of any function y of x is defined, as in § 138, to be the *limit of the quotient* $\Delta y/\Delta x$ *as* Δx *approaches zero,* Δy being the increment of the function y corresponding to the increment Δx of x; in symbols:

$$y' = \lim_{\Delta x = 0} \frac{\Delta y}{\Delta x}.$$

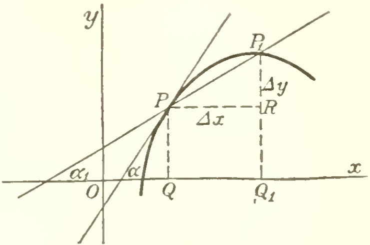


FIG. 57

Geometrically this means that y' is the slope of the tangent of the curve whose ordinate is y. For, $\Delta y/\Delta x$ is the *slope of the secant* PP_1 (Fig. 57):

$$\frac{\Delta y}{\Delta x} = \tan \alpha_1;$$

and the limit of this quotient as Δx approaches zero, *i.e.* as P_1 moves along the curve to P, is the *slope of the tangent* at P:

$$y' = \tan \alpha = \lim_{\Delta x = 0} \frac{\Delta y}{\Delta x}.$$

If the function y be denoted by $f(x)$, then

$$\Delta y = f(x + \Delta x) - f(x);$$

hence

$$y' = \lim_{\Delta x = 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

150. Calculation of the Derivative. To find, by means of the last formula, the derivative of the polynomial

$$y = f(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n,$$

we should have to form first $f(x + \Delta x)$, *i.e.*

$$(x + \Delta x)^n + a_1 (x + \Delta x)^{n-1} + \cdots + a_n,$$

subtract from this the original polynomial, then divide by Δx, and finally put $\Delta x = 0$.

This rather cumbersome process can be avoided if we observe that a polynomial is a sum of terms of the form ax^n and apply the following fundamental propositions about derivatives:

(1) *the derivative of a sum of terms is the sum of the derivatives of the terms;*

(2) *the derivative of* ax^n *is* a *times the derivative of* x^n*;*

(3) *the derivative of a constant is zero;*

(4) *the derivative of* x^n *is* nx^{n-1}.

The first three of these propositions can be regarded as obvious; a fuller discussion of them, based on an exact definition of the limit of a function, is given in the differential calculus. A proof of the fourth proposition is given in the next article.

On the basis of these propositions we find at once that the derivative of the polynomial

$$y = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n$$

is

$$y' = a_0nx^{n-1} + a_1(n-1)x^{n-2} + a_2(n-2)x^{n-3} + \cdots + a_{n-1}.$$

151. Derivative of x^n. By the definition of the derivative (§ 149) we have for the derivative of $y = x^n$:

$$y' = \lim_{\Delta x = 0} \frac{(x + \Delta x)^n - x^n}{\Delta x}.$$

Now by the binomial theorem (see below, § 152) we have

$$(x + \Delta x)^n = x^n + nx^{n-1}\Delta x + \frac{n(n-1)}{1 \cdot 2}x^{n-2}(\Delta x)^2 + \cdots + (\Delta x)^n,$$

and hence

$$(x + \Delta x)^n - x^n = nx^{n-1}\Delta x + \frac{n(n-1)}{1 \cdot 2}x^{n-2}(\Delta x)^2 + \cdots + (\Delta x)^n.$$

Dividing by Δx and then letting Δx become zero, we find

$$y' = nx^{n-1}.$$

EXERCISES

1. Find the derivatives of the following functions of x by means of the fundamental definition (§ 149) and check by § 150:

(*a*) x^3.
(*b*) $x^2 + x$.
(*c*) $x^4 + 6x^2$.
(*d*) $-6x^3$.
(*e*) $x^4 - 3x^3$.
(*f*) $mx + b$.

2. Find the derivatives of the following functions:

(*a*) $5x^4 - 3x^2 + 6x$.
(*b*) $1 - x + \frac{1}{2}x^2 - \frac{1}{3}x^3$.
(*c*) $(x - 2)^3$.
(*d*) $(2x + 3)^5$.
(*e*) $3(4x - 1)^3$.
(*f*) $x^n + ax^{n-1} + bx^{n-2}$.

3. For the following functions write the derivative indicated:

(*a*) $5x^3 - 3x$, find y'''.
(*b*) $ax^2 + bx + c$, find y'''.
(*c*) x^5, find y^{v}.
(*d*) $ax^3 + bx^2 + cx + d$, find y^{iv}.
(*e*) $\frac{1}{6}x^6$, find y''.
(*f*) $\dfrac{1}{6\,!}x^6$, find y^{vii}.
(*g*) $x^{12} - qx^8$, find y'''.
(*h*) $(2x - 3)^3$, find y'''.

152. Binomial Theorem. In § 151 we have used the binomial theorem for a positive integral exponent n, *i.e.* the proposition that

$$(1) \qquad (x + h)^n = x^n + nx^{n-1}h + \frac{n(n-1)}{1 \cdot 2}x^{n-2}h^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}x^{n-3}h^3 + \cdots + \frac{n(n-1)\cdots 1}{n\,!}h^n.$$

The formula (1) can be proved by mathematical induction (§ 62). It certainly holds for $n = 2$, since by direct multiplication we have

$$(x + h)^2 = x^2 + 2xh + h^2 = x^2 + 2xh + \frac{2 \cdot 1}{2\,!}h^2,$$

which agrees with (1) for $n = 2$.

Moreover, if the formula (1) holds for any exponent n, it holds for $n + 1$. For, multiplying (1) by $x + h$ in both members, we find

$$(x + h)^{n+1} = x^{n+1} + (n + 1)x^n h + \frac{(n+1)n}{1 \cdot 2}x^{n-1}h^2 + \cdots + \frac{(n+1)n(n-1)\cdots 1}{(n+1)!}h^{n+1},$$

which is the form that (1) assumes when n is replaced by $n + 1$.

153. Binomial Coefficients. The coefficients

$$1,\ n,\ \frac{n(n-1)}{2\,!},\ \cdots\ \frac{n(n-1)\cdots 2}{(n-1)\,!} = n,\quad \frac{n(n-1)\cdots 1}{n\,!} = 1$$

in the binomial formula (1) are called the *binomial coefficients*.

The meaning of these coefficients will appear from another proof of the formula, which is as follows: If n is a positive integer, we can write $(x+h)^n$ in the form

$$(x+h_1)(x+h_2)(x+h_3)(x+h_4)\cdots(x+h_n),$$

where the subscripts are used simply for convenience to distinguish the binomial factors: *i.e.* it is understood that $h_1 = h_2 = h_3 = \cdots = h_n = h$. Each term in the expanded product is the product of n letters of which one and only one is taken from each binomial factor. To form all these terms we may proceed as follows:

(a) If we choose x from each of the n factors, we obtain as first term of the expansion x^n.

(b) If we choose x from $n-1$ factors, the letter h can be chosen from any one of the n factors, *i.e.* it can be chosen in ${}_nC_1$ ways (§ 64); this gives

$$x^{n-1}(h_1 + h_2 + \cdots + h_n), \text{ the number of terms being } {}_nC_1.$$

(c) If we choose x from $n-2$ factors, the other two letters can be chosen from any two of the n factors, *i.e.* in ${}_nC_2$ ways; this gives

$$x^{n-2}(h_1h_2 + h_1h_3 + \cdots + h_2h_3 + \cdots), \text{ the number of terms being } {}_nC_2.$$

(d) If we choose x from $n-3$ factors, the other three letters can be chosen from any three of the n factors, *i.e.* in ${}_nC_3$ ways; this gives

$$x^{n-3}(h_1h_2h_3 + h_1h_2h_4 + \cdots + h_2h_3h_4 + \cdots), \text{ the number of terms being } {}_nC_3.$$

Finally we have to choose no x and consequently an h from every factor, which can be done in ${}_nC_n = 1$ way; this gives the last term

$$h_1h_2\cdots h_n.$$

Now as $h_1 = h_2 = \cdots = h_n = h$, we find the binomial expansion:

$$(x+h)^n = x^n + {}_nC_1x^{n-1}h + {}_nC_2x^{n-2}h^2 + \cdots + {}_nC_{n-1}xh^{n-1} + {}_nC_nh^n.$$

Since, by § 64,

$${}_nC_1 = n, \quad {}_nC_2 = \frac{n(n-1)}{1\cdot 2}, \quad \cdots \quad {}_nC_{n-1} = n, \quad {}_nC_n = 1$$

this form agrees with that of § 152. It will now be clear why the binomial coefficients are the numbers of combinations of n elements, 1, 2, 3, $\cdots$ at a time.

The proof also shows that the binomial coefficients are equal in pairs, the first being equal to the last, the second to the last but one, etc.

Finally it may be noted that, with $x = 1$, $h = 1$ we obtain the following remarkable expression for the *sum of the binomial coefficients*:

$$2^n = 1 + {}_nC_1 + {}_nC_2 + \cdots + {}_nC_n.$$

EXERCISES

1. Show that in the expansion of $(x-h)^n$ by the binomial theorem the signs of the terms are alternately + and −.

2. If the binomial coefficients of the first, second, third, fourth, etc., power of a binomial are written down as in the horizontal lines of the adjoining diagram, it will be observed that (excepting the ones) every figure is the sum of the two just above it. Extend the triangle by this rule to the 10th power, and prove the rule (see § 152).

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
. . . .

PASCAL'S TRIANGLE

3. Expand by the binomial theorem:

(*a*) $(x + 2y)^3$. (*b*) $\left(x^2 + \frac{3}{x}\right)^4$. (*c*) $(2a - c)^3$.

(*d*) $\left(\frac{x}{y} - \frac{y^2}{x^2}\right)^4$. (*e*) $(a + b + c)^3$. (*f*) $(4x - \frac{1}{2}y)^3$.

(*g*) $(1 + 2x)^3 - (1 - 2x)^3$. (*h*) $(1 - x)^{10}$. (*i*) $\left(x - \frac{1}{x}\right)^6$.

(*j*) $\left(\frac{x}{2} - \frac{1}{x} - 1\right)^3$. (*k*) $(\frac{3}{2} - x^2)^4$. (*l*) $(a + b - c - d)^3$.

4. Write the term indicated:

(*a*) Fourth term in $(a + b)^{12}$.
(*b*) Fourth term in $(a - b)^{12}$.
(*c*) Tenth term in $(x^2 + 4y^2)^{15}$.
(*d*) Middle term in $(x^{\frac{1}{2}} - y^{\frac{1}{2}})^{10}$.
(*e*) kth term in $(x + h)^n$.
(*f*) kth term in $(x - h)^n$.
(*g*) Two middle terms in $(a^2 - 2b^2)^5$.
(*h*) Term next to the last in $\left(a - \frac{1}{a}\right)^{27}$.

5. Show that the sum of the coefficients in the expansion of $(x-h)^n$ is zero when n is an odd integer.

6. Use the binomial formula to find (*a*) $(1.02)^5$; (*b*) $(3.97)^4$.

154. Properties of the General Polynomial Curve. In plotting the curve

$$y = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n$$

observe that (Fig. 58):

(*a*) the intercept OB on the axis Oy is equal to the constant term a_n:

(*b*) the intercepts OA_1, OA_2, $\cdots$ on the axis Ox are roots of the equation $y = 0$, *i.e.*

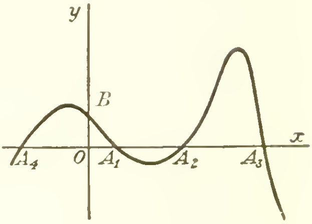


FIG. 58

$$a_0x^n + a_1x^{n-1} + \cdots + a_n = 0;$$

(c) the abscissas of the least and greatest ordinates are found by solving the equation $y' = 0$, *i.e.* (§ 150)

$$na_0x^{n-1} + \cdots + a_{n-1} = 0,$$

every real root giving a minimum ordinate if for this root y'' is positive and a maximum ordinate if y'' is negative;

(*d*) the abscissas of the points of inflection are found by solving the equation $y'' = 0$, *i.e.*

$$n(n-1)a_0x^{n-2} + \cdots + 2\,a_{n-2} = 0,$$

every real root of this equation being the abscissa of a point of inflection provided that $y''' \neq 0$. (If y''' were zero, y' might not be a maximum or minimum, and further investigation would be necessary.)

155. Continuity of Polynomials. It should also be observed that the function $y = a_0x^n + a_1x^{n-1} + \cdots + a_n$ is one-valued, real, and finite for every x; *i.e.* to every real and finite abscissa x belongs one and only one ordinate, and this ordinate is real and finite. Moreover, as the first derivative $y' = na_0x^{n-1} + \cdots + a_{n-1}$ is again a polynomial, the slope of the curve is everywhere one-valued and finite.

Thus, so-called *discontinuities* of the ordinate (Fig. 59) or of the slope (Fig. 60) cannot occur: the curve $y = a_0x^n + \cdots + a_n$ is *continuous.*

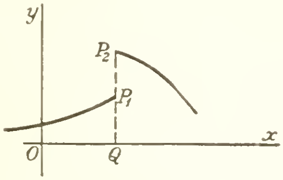


FIG. 59

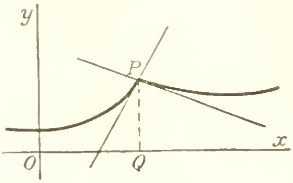


FIG. 60

Strictly defined, the **continuity** of the function $y = a_0x^n + \cdots + a_n$ means that, for every value of x, *the limit of the function is equal to the value of the function.* The function $y = a_0x^n + \cdots + a_n$ has one and only one value for any value $x = x_1$, viz. $a_0x_1^n + \cdots + a_n$. The value of the function for any other value of x, say for $x_1 + \Delta x$, is $a_0(x_1 + \Delta x)^n + \cdots + a_n$ which can be written in the form $a_0x_1^n + \cdots + a_n +$ terms containing Δx as factor. Therefore as Δx approaches zero, the function approaches a limit, viz. its value for $x = x_1$.

156. Intermediate Values. A continuous function, in varying from any value to any other value, must necessarily pass through all intermediate values. Thus, our polynomial $y = a_0x^n + \cdots + a_n$, if it passes from a negative to a positive value (or *vice versa*), must pass through zero. It follows from this that *between any two ordinates of opposite sign the curve $y = a_0x^n + \cdots + a_n$ must cross the axis Ox at least once.*

It also follows from the continuity of the polynomial and its derivatives that *between any two intersections with the axis Ox there must lie at least one maximum or minimum,* and between a maximum and a minimum there must lie a point of inflection.

Ordinates at particular points can be calculated by the process of § 148.

EXERCISES

1. Sketch the following curves:

(*a*) $y=(x-1)(x-2)(x-3)$. (*b*) $4y = x^4-1$. (*c*) $10y = x^5$.

(*d*) $10y = x^5 + 5$. (*e*) $4y=(x+2)^2(x-3)$. (*f*) $y=(x-1)^4$.

2. When is the curve $y = a_0x^n + a_1x^{n-1} + \cdots + a_n$ symmetric with respect to Oy?

3. Determine the coefficients so that the curve $y = a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4$ shall touch Ox at $(1, 0)$ and at $(-1, 0)$ and pass through $(0, 1)$, and sketch the curve.

4. Find the coordinates of the maxima, minima, and points of inflection and then sketch the curve $4y = x^4 - 2x^2$.

5. Are the following curves concave upward or downward at the indicated points?

(*a*) $16y = 16x^4 - 8x^2 + 1$, at $x = -1, -\frac{1}{2}, 0, \frac{1}{2}, 3$.

(*b*) $y = 4x - x^4$, at $x = -2, 0, 1, 3$.

(*c*) $y = x^n$, at any point; distinguish the cases when n is a positive even or odd integer.

6. What happens to the curves $y = ax^3$ and $y = ax^4$ as a changes? For example, take $a = 2, 1, \frac{1}{3}, 0, -\frac{1}{3}, -1, -2$.

7. Find the values of x for which the following relations are true:

(*a*) $x^4 - 6x^2 + 9 \gtreqless 0$. (*b*) $(x-1)^2(x^2-4) \gtreqless 0$.

8. Show that the following curves do not cross the axis Ox outside of the intervals indicated:

(*a*) $y = x^3 - 2x^2 + 4x + 5$, between -2 and 2.

(*b*) $y = x^4 - 5x^2 + 6x - 3$, -3 and 3.

(*c*) $y = x^3 - x^2 + 3x - 3$, 0 and 1.

(*d*) $y = x^4 + x^2 - 3x + 2$, 0 and 1.

9. Those curves whose ordinates represent the values of the first, second, etc., derivatives of a given polynomial are called the first, second, etc., *derived curves*. Sketch on the same coordinate axes the following curves and their derived curves:

(*a*) $6y = 2x^3 - 3x^2 - 12x$. (*b*) $y = (x-2)^2(x+1)$.

(*c*) $y = (x+1)^3$. (*d*) $2y = x^4 + x^2 + 1$.

10. At what point on Ox must the origin be taken in order that the equation of the curve $y = 2x^3 - 3x^2 - 12x - 5$ shall have no term in x^2? no term in x?

PART IV. NUMERICAL EQUATIONS

157. Equations. Roots. In plotting the curves $y = a_0x^n + \cdots + a_n$ (§ 154) it is often desirable to solve equations of the form

$$(1) \qquad a_0x^n + \cdots + a_n = 0,$$

the coefficients a_0, a_1, $\cdots$ a_n being given real numbers and n any positive integer. The solution of such ***numerical equations***, at least approximately, presents itself in many other problems. The roots of the equation (1) are also called the *roots*, or *zeros, of the function* $a_0x^n + \cdots + a_n$.

It is understood that $a_0 \neq 0$ since otherwise the equation would not be of degree n. We can therefore divide (1) by a_0 and write the equation in the form

$$(2) \qquad x^n + p_1x^{n-1} + \cdots + p_n = 0,$$

where $p_1 = a_1/a_0$, $p_2 = a_2/a_0$, $\cdots$ $p_n = a_n/a_0$ are given real numbers.

158. Relation of Coefficients to Roots. We here assume the fundamental theorem of algebra that every equation of the form (2) has at least one root, say $x = x_1$, which may be real or imaginary. If we then divide the polynomial $x^n + p_1x^{n-1} + \cdots + p_n$ by $x - x_1$, we obtain a polynomial of degree $n - 1$; the equation of the $(n-1)$th degree obtained by equating this polynomial to zero must again have at least one root. Proceeding in this way, we find that *every equation of the form* (2) *has n roots*, which of course may be real or complex, and some of which may be equal. It also appears that the equation (2) may be written in the form

$$(3) \qquad (x - x_1)(x - x_2) \cdots (x - x_n) = 0,$$

where x_1, x_2, $\cdots$ x_n are the n roots, or performing the multiplication (§ 153):

$$(4) \quad x^n - (x_1 + \cdots + x_n)x^{n-1} + (x_1x_2 + \cdots + x_{n-1}x_n)x^{n-2} + \cdots + (-1)^n x_1 \cdots x_n = 0.$$

Comparing the coefficients in (4) with those in (2), we find:

$$x_1 + \cdots + x_n = -p_1,$$
$$x_1x_2 + \cdots + x_{n-1}x_n = p_2,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$x_1x_2 \cdots x_n = (-1)^n p_n;$$

i.e. if the coefficient of the highest power of a polynomial is one, then *the coefficient of* x^{n-1}, *with sign reversed, is equal to the sum of the roots;* the coefficient of x^{n-2} is equal to the sum of the products of the roots two at a time; minus the coefficient of x^{n-3} is equal to the sum of the products of the roots three at a time, etc.; *plus or minus the constant term* (according as n is even or odd) *is equal to the product of all the roots.*

159. Equations with Integral Coefficients. The results of the last article can often be used to advantage to find the roots of a numerical equation (2) in which all the coefficients $p_1, \cdots p_n$ are integers. We then try to resolve the left-hand member into linear factors of the form $x - x_k$; if this can be done, the roots are the numbers x_k.

The fact that the constant term p_n in (2) is plus or minus the product of the roots can be used in the same case by trying to see whether any one of the integral factors of $\pm p_n$ satisfies the equation.

EXERCISES

1. Find the roots of: (*a*) $x^2 - 7x + 6 = 0$; (*b*) $x^3 - 2x^2 - 13x - 10 = 0$; (*c*) $x^4 - 1 = 0$; (*d*) $x^4 - 7x^2 - 18 = 0$; (*e*) $x^3 - 5x^2 - 2x + 24 = 0$.

2. Form the equation whose roots are: (*a*) $2, -2, 3$; (*b*) $-1, -1, 1$; (*c*) $0, \sqrt{2}, -\sqrt{2}$; (*d*) $-1, 1, \frac{1}{2}, -\frac{3}{2}$.

3. For the equation $x^3 + p_1x^2 + p_2x + p_3 = 0$ determine the relation between the coefficients when: (*a*) two roots are equal but opposite in sign; (*b*) the product of two roots is equal to the square of the third; (*c*) the three roots are equal.

4. Show that the sum of the n nth roots of any number is zero. What about the sum of the products of the roots two at a time? three at a time?

160. Imaginary Roots. In general, the real roots of a numerical equation are of course not integers, nor even rational fractions, but irrational numbers. In solving such an equation the object is to find a number of decimal places of each root sufficient for the problem in hand. Methods of approximation appropriate for this purpose are given in the following articles.

The ***imaginary roots*** of the equation can be determined by somewhat similar, though more laborious, processes. It will here suffice to show that *imaginary roots always occur in pairs of conjugates ;* that is, *if an imaginary number* $\alpha+\beta i$ *is a root of the equation* (*1*) (with real coefficients), *then the conjugate imaginary number* $\alpha-\beta i$ *is a root of the same equation.*

For, substituting $\alpha+\beta i$ for x in (1) and collecting the real and pure imaginary terms separately, we obtain an equation of the form

$$A+Bi=0,$$

where A and B are real; hence, by § 116, $A=0$ and $B=0$.

If, on the other hand, we substitute in (1) $\alpha-\beta i$ for x, the result must be the same except that i is replaced by $-i$: we find therefore $A-Bi=0$, and this is satisfied if $A=0$ and $B=0$, *i.e.* if $\alpha+\beta i$ is a root.

It follows in particular that *a cubic equation always has at least one real root.* Indeed, in the case of the cubic equation, only two cases are possible: (a) the equation has three real roots, which may of course be all different, or two equal but different from the third, or all three equal; (b) the equation has one real and two conjugate imaginary roots.

161. Methods of Approximation for Real Roots. If a good sketch of the curve $y=a_0x^n+\cdots+a_n$ were given, we could obtain approximate values of the real roots of the equation

$$a_0x^n+\cdots+a_n=0$$

by measuring the intercepts OA_1, OA_2, etc., made by the curve on the axis Ox (§ 154). If the curve is not given, we calculate a number of ordinates for various values of x until we find two ordinates of opposite sign; we know (§ 156) that the curve must cross the axis Ox between these ordinates, and therefore at least one real root of the equation must lie between the abscissas, say x_1 and x_2, whose ordinates are of opposite sign.

We can next contract the interval between which the root lies by calculating intermediate ordinates. By this process a root can be calculated to any desired degree of accuracy. But the process is rather long and laborious. The calculation of the ordinates is best performed by the process of § 148.

162. Interpolation. If the interval within which the root has been confined is small, we can obtain, without calculating further ordinates, a further approximation to the root by replacing the curve in the interval by its secant, and finding its intersection with the axis Ox.

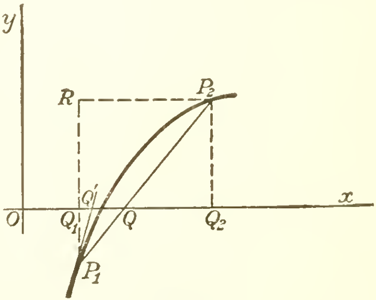


FIG. 61

Suppose (Fig. 61) that we have found that a root lies between $OQ_1 = x_1$ and $OQ_2 = x_2$, the ordinates $Q_1P_1 = y_1$ and $Q_2P_2 = y_2$ being of opposite sign. Then x_1 is a *first approximation* to the root x; and if Q_1 and Q_2 lie close together, the intercept OQ made by the secant P_1P_2 on the axis Ox is a *second approximation*. Let us calculate the **correction** $Q_1Q = h$ which must be added to the first approximation x_1 to obtain the second approximation $x_1 + h$.

The figure shows that $Q_1Q/RP_2 = P_1Q_1/P_1R$, *i.e.*

$$\frac{h}{x_2 - x_1} = \frac{-y_1}{y_2 - y_1};$$

hence the correction h is

$$h = -\frac{x_2 - x_1}{y_2 - y_1} y_1 = -\frac{\Delta x}{\Delta y} y_1.$$

This process, which is the same as that used in interpolating in a table of logarithms, is known as the *regula falsi*, or rule of false position.

163. Tangent Method. Another method for finding a correction consists in using the intercept made on the axis Ox not by the secant but by the *tangent* to the curve at P_1.

The correction $Q_1Q' = k$ is found (Fig. 61) from the triangle P_1Q_1Q', in which the tangent of the angle at Q' is equal to the value of the derivative y_1' at P_1. This triangle gives

$$y_1' = \frac{P_1Q_1}{k} = -\frac{y_1}{k};$$

hence
$$k = -\frac{y_1}{y_1'}.$$

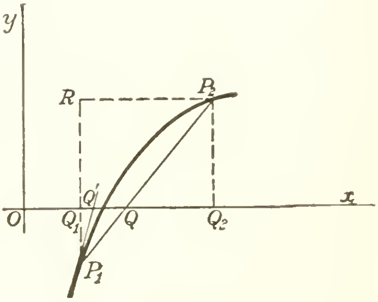


FIG. 61

Find by this method the roots of $x^3 - 3x + 1 = 0$.

164. Newton's Method of Approximation. After finding, by § 161, a first approximation x_1 to a root of the equation

$$a_0x^n + a_1x^{n-1} + \cdots + a_n = 0, \tag{1}$$

transfer the origin to the point $(x_1, 0)$. Thus (Fig. 62), if a root lies between 3 and 4, transform the equation to $(3, 0)$ as origin, by replacing x by $3 + h$. An expeditious process for finding the new equation in h, say

$$b_0h^n + b_1h^{n-1} + \cdots + b_n = 0, \tag{2}$$

will be given in §§ 165–167.

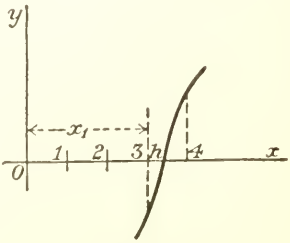


FIG. 62

As h is a proper fraction, its higher powers will be small, so that an *approximate* value of h can be obtained from the linear terms, *i.e.* by solving $b_{n-1}h+b_n=0$, which gives h approximately $=-b_n/b_{n-1}$. Hence we put

$$h=-\frac{b_n}{b_{n-1}}+k, \tag{3}$$

where k is a still smaller proper fraction. If the approximation obtained from the linear terms should be too rough, we may find a better approximation of h by solving the quadratic

$$b_{n-2}h^2+b_{n-1}h+b_n=0.$$

We next substitute the value (3) of h in (2) and proceed in the same way with the equation in k. The process can be repeated as often as desired; the last division can be carried to about as many more significant figures as have been obtained before. The example in § 168 will best explain the work.

165. Remainder Theorem. If a polynomial $f(x)=a_0x^n+a_1x^{n-1}+\cdots+a_n$ of degree n be divided by $x-h$, there is obtained in general a quotient Q, which is a polynomial of degree $n-1$, and a remainder R:

$$\frac{f(x)}{x-h}=Q+\frac{R}{x-h}, \text{ i.e. } f(x)=Q(x-h)+R.$$

For $x=h$ the last equation gives $f(h)=R$; i.e. *the value of the polynomial for any particular value h of x is equal to the remainder R obtained upon dividing the polynomial by $x-h$:*

$$f(h)=a_0h^n+\cdots+a_n=R.$$

This proposition is known as the *remainder theorem.*

166. Synthetic Division. As an example let us divide

$$f(x)=2x^3-3x^2-12x+5$$

by $x-3$. By any method we obtain the following result:

$$\frac{2x^3-3x^2-12x+5}{x-3}=2x^2+3x-3-\frac{4}{x-3}.$$

The elementary method is as follows:

$$\begin{array}{llll}
2x^3 - 3x^2 - 12x + 5 & \multicolumn{3}{l}{\underline{x-3} = 2x^2 + 3x - 3} \\
\underline{2x^3 - 6x^2} & & & \\
\quad\quad 3x^2 - 12x & & & \\
\quad\quad \underline{3x^2 - \ \ 9x} & & & \\
\quad\quad\quad\quad -3x + 5 & & & \\
\quad\quad\quad\quad \underline{-3x + 9} & & & \\
\quad\quad\quad\quad\quad\quad -4 & & &
\end{array}$$

This process can be notably shortened:

(*a*) As the dividend is a polynomial, it can be indicated sufficiently by writing down its coefficients only, any missing term being supplied by a zero:

$$2 \quad -3 \quad -12 \quad 5$$

(*b*) As x in the divisor has the coefficient 1, the first terms of the partial products need not be written; the second terms it is more convenient to change in sign; in other words, instead of multiplying by -3 and subtracting, multiply by $+3$ and add.

The whole calculation then reduces to the following scheme:

$$\begin{array}{rrrr|l}
2 & -3 & -12 & 5 & \underline{3} \\
 & 6 & 9 & -9 & \\
\hline
2 & 3 & -3 & -4 &
\end{array}$$

This is the same scheme as that in § 148. But it should be observed that this method, known as ***synthetic division***, gives not only the remainder -4, *i.e.* $f(3)$, but also the coefficients $2, 3, -3$ of the quotient.

167. Calculation of $f(x_1 + h)$. If in $f(x) = a_0x^n + \cdots + a_n$ we substitute $x = x_1 + h$, we find:

$$f(x) = f(x_1 + h) = a_0(x_1 + h)^n + a_1(x_1 + h)^{n-1} + \cdots + a_{n-1}(x_1 + h) + a_n.$$

Expanding the powers of $x_1 + h$ by the binomial theorem and arranging in descending powers of h we obtain a result of the form

$$f(x) = f(x_1 + h) = b_0h^n + b_1h^{n-1} + \cdots + b_{n-1}h + b_n.$$

To find the coefficients $b_0, b_1, \cdots b_n$ of this expansion of $f(x_1 + h)$ in powers of h observe that as $h = x - x_1$ we have

$$f(x) = f(x_1 + h) = b_0(x - x_1)^n + b_1(x - x_1)^{n-1} + \cdots + b_{n-1}(x - x_1) + b_n.$$

The last term, b_n, is therefore the remainder obtained upon dividing $f(x)$ by $x - x_1$; it is best found by synthetic division (§ 166). The quotient obtained upon dividing $f(x)$ by $x - x_1$ is evidently $b_0(x - x_1)^{n-1} + b_1(x - x_1)^{n-2} + \cdots + b_{n-1}$; the last term, b_{n-1}, can again be obtained as the remainder upon division by $x - x_1$. Proceeding in this way all the coefficients $b_n, b_{n-1}, \cdots b_1, b_0$ can be found.

For the example of § 166 we have

$$\begin{array}{rrrr|l}
2 & -3 & -12 & 5 & 3 \\
 & 6 & 9 & -9 & \\
\hline
2 & 3 & -3 & \mathbf{-4} & \\
 & 6 & 27 & & \\
\hline
2 & 9 & \mathbf{24} & & \\
 & 6 & & & \\
\hline
\mathbf{2} & \mathbf{15} & & &
\end{array}$$

The result is: $f(3 + h) = 2h^3 + 15h^2 + 24h - 4.$

168. Example. The roots of the equation

$$2x^3 - 3x^2 - 12x + 5 = 0$$

are readily found to lie between -3 and -2, 0 and 1, 3 and 4. To calculate the last of these we find by transferring the origin to the point $(3, 0)$ the following equation for the correction h to the first approximation, which is 3 (§ 167):

$$2h^3 + 15h^2 + 24h - 4 = 0.$$

The linear terms give $h = 1/6 = 0.17$; as the quadratic term, $15h^2$, is about 0.42 and 1/24 of this is 0.02, a somewhat better approximation is $h = 0.15$. Substituting

$$h = 0.15 + h_1,$$

we find:

$$\begin{array}{rrrr}
2 & 15 & 24 & -4 \\
 & 0.30 & 2.295 & 3.94425 \\
\hline
2 & 15.30 & 26.295 & -0.05575 \\
 & .30 & 2.340 & \\
\hline
2 & 15.60 & 28.635 & \\
 & .30 & & \\
\hline
2 & 15.90 & &
\end{array}$$

Hence the equation for h_1 is

$$2h_1^3 + 15.90h_1^2 + 28.635h_1 - 0.05575 = 0.$$

The linear terms give $h_1 = 0.001947.$

As the quadratic term can influence only the 6th decimal place, we can certainly take $h_1 = 0.00195$ and thus find the root 3.15195.

169. Negative Roots. To find a negative root replace x by $-x$ in the given equation. *i.e.* reflect the curve in the axis Oy.

To find a root greater than 10 replace x by $10z$, or $100z$, etc., in the given equation, and calculate z.

170. Horner's Process. W. G. Horner's method is essentially the same as Newton's, inasmuch as it consists in moving the origin closer and closer up to the root. But it calculates each significant figure separately. Thus, for the example of § 168 we should proceed as follows:

As in §§ 167, 168, we diminish the roots of the equation

$$2x^3 - 3x^2 - 12x + 5 = 0$$

by 3 so that the equation (as there shown) takes the form

$$2x^3 + 15x^2 + 24x - 4 = 0.$$

The left-hand member changes sign between 0.1 and 0.2. We move therefore the origin through 0.1 to the right:

$$\begin{array}{rrrr}
2 & 15 & 24 & -4 \\
 & .2 & 1.52 & 2.552 \\
\hline
2 & 15.2 & 25.52 & -1.448 \\
 & .2 & 1.54 & \\
\cline{1-3}
2 & 15.4 & 27.06 & \\
 & .2 & & \\
\cline{1-2}
2 & 15.6 & &
\end{array}$$

The new equation is $2x^3 + 15.6x^2 + 27.06x - 1.448 = 0$.

The left-hand member changes sign between 0.05 and 0.06; hence we move the origin through 0.05:

$$\begin{array}{rrrr}
2 & 15.6 & 27.06 & -1.448 \\
 & .10 & .785 & 1.39225 \\
\hline
2 & 15.70 & 27.845 & -0.05575 \\
 & .10 & .790 & \\
\cline{1-3}
2 & 15.80 & 28.635 & \\
 & .10 & & \\
\cline{1-2}
2 & 15.90 & &
\end{array}$$

The new equation is $2x^3 + 15.90x^2 + 28.635x - 0.05575 = 0$.

We can evidently go on in the same way finding more decimal places. It should not be forgotten (§ 164) that after finding a number of significant figures in this way, about as many more can be found by simple division. Thus, we have found $x = 3.15 \cdots$; the linear terms of the last equation give the correction 0.00195, so that $x = 3.15195$.

EXERCISES

1. Find: (*a*) the cube root of 67; (*b*) the fourth root of 19; (*c*) the fifth root of 7, to seven significant figures, and check by logarithms.

2. Newton used his method to approximate the positive root of $x^3 - 2x - 5 = 0$; find this root to eight significant figures.

3. Find, to five significant figures, the root of the equation

$$x^3 + 2.73\,x^2 = 0.375.$$

4. Find the coordinates of the intersections of the curve $y = (x-1)^2(x+2)$ with the lines: (*a*) $y = 3$; (*b*) $y = \frac{1}{3}x + 1$; (*c*) $y = \frac{1}{4}x - \frac{1}{2}$.

5. After cutting off slices of thickness 1 in., 1 in., 2 in., parallel to three perpendicular faces of a cube, the volume is 8 cu. in. What was the length of an edge of the cube?

6. Find the radius of that sphere whose volume is decreased 50 % when the radius is decreased 2 ft.

7. For what values of k will the lines $kx + y + 2 = 0$, $x + ky - 1 = 0$, $2x - y + k = 0$ pass through a common point?

8. For what values of k are the following equations satisfied by other values of x, y, z, w than 0, 0, 0, 0? $kx + 2y + z - 3w = 0$, $2x + ky + z - w = 0$, $x - 2y + kz + w = 0$, $x + 7y - z + kw = 0$.

9. A buoy composed of a cone of altitude 6 ft. surmounted by a hemisphere with the same base when submerged displaces a volume of water equal to a sphere of radius 5 ft. Find the radius of the buoy.

10. Find, to four significant figures, the coordinates of the intersections of the parabolas $y + x^2 = 7$, $x + y^2 = 11$, Ex. 13, p. 138.

11. By applying Newton's method (§ 164) to both coordinate axes simultaneously, find that intersection of the parabolas $x^2 - y = 4$ and $x + y^2 = 3$ which lies in the first quadrant.

12. The segment cut out of a sphere of radius a by a plane through its center and a parallel plane at the distance x from it has a volume $= \pi x(a^2 - \frac{1}{3}x^2)$; at what distance from its base must a hemisphere be cut by a plane parallel to the base to bisect the volume of the hemisphere?

171. Expansion of $f(x+h)$. The solution of numerical equations is based on the fundamental fact (§ 167) that if $f(x)$ is a polynomial, then $f(x_1+h)$ can be expressed as a polynomial of the same degree in h, and the coefficients A_0, A_1, $\cdots A_n$ of this polynomial can be calculated. Thus, for

$$f(x)=a_0x^4+a_1x^3+a_2x^2+a_3x+a_4$$

we have:

$$\begin{aligned} f(x_1+h) &= a_0(x_1+h)^4+a_1(x_1+h)^3+a_2(x_1+h)^2+a_3(x_1+h)+a_4 \\ &= a_0x_1^4+a_1x_1^3+a_2x_1^2+a_3x_1+a_4 \\ &\quad +(4\,a_0x_1^3+3\,a_1x_1^2+2\,a_2x_1+a_3)h \\ &\quad +(6\,a_0x_1^2+3\,a_1x_1+a_2)h^2 \\ &\quad +(4\,a_0x_1+a_1)h^3 \\ &\quad +a_0h^4. \end{aligned}$$

Now this process is closely connected with that of finding the successive derivatives of the polynomial. Thus we have for

$$f(x)=a_0x^4+a_1x^3+a_2x^2+a_3x+a_4$$

the derivatives:

$$\begin{aligned} f'(x) &= 4\,a_0x^3+3\,a_1x^2+2\,a_2x+a_3, \\ f''(x) &= 12\,a_0x^2+6\,a_1x+2\,a_2, \\ f'''(x) &= 24\,a_0x+6\,a_1, \\ f^{\text{iv}}(x) &= 24\,a_0, \end{aligned}$$

all higher derivatives being zero. If in these expressions we put $x=x_1$ and then multiply them respectively by 1, h, $h^2/2\,!$, $h^3/3\,!$, $h^4/4\,!$, and add, we find precisely the above expression for $f(x_1+h)$; hence we have:

$$f(x_1+h)=f(x_1)+f'(x_1)h+\frac{f''(x_1)}{1\cdot 2}h^2+\frac{f'''(x_1)}{1\cdot 2\cdot 3}h^3+\frac{f^{\text{iv}}(x_1)}{1\cdot 2\cdot 3\cdot 4}h^4,$$

whenever $f(x)$ is a polynomial of degree 4.

It can be proved in the same way that for a polynomial of degree n we have

$$f(x_1+h)=f(x_1)+f'(x_1)h+\frac{f''(x_1)}{1\cdot 2}h^2+\cdots+\frac{f^{(n)}(x_1)}{n\,!}h^n.$$

This formula is a particular case of a general proposition of the differential calculus, known as ***Taylor's theorem***. It shows that *the value of a polynomial for any value $x=x_1+h$ can be found if we know the value of the polynomial itself and of all its n derivatives for some particular value x_1 of x.* This property is characteristic for polynomials.

CHAPTER IX

THE PARABOLA

172. The Parabola. The ***parabola*** can be defined as the *locus of a point whose distance from a fixed point is equal to its distance from a fixed line.* The fixed point is called the ***focus***, the fixed line the ***directrix***, of the parabola.

Let F (Fig. 63) be the fixed point, d the fixed line; then every point P of the parabola must satisfy the condition

$$FP = PQ,$$

Q being the foot of the perpendicular from P to d. Let us take F as origin, or pole, and the perpendicular FD from F to the directrix as polar axis, and let the given distance $FD = 2a$. Then $FP = r$ and $PQ = 2a - r\cos\phi$. The condition $FP = PQ$ becomes therefore

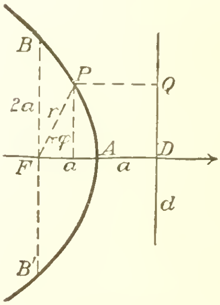


FIG. 63

i.e. $$r = 2a - r\cos\phi,$$

(1) $$r = \frac{2a}{1 + \cos\phi}.$$

This equation, which expresses the radius vector of P as a function of the vectorial angle ϕ, is *the* ***polar equation*** *of the parabola, when the focus is taken as pole and the perpendicular from the focus to the directrix as polar axis.*

173. Polar Construction of Parabolas. By means of the equation (1) the parabola can be plotted by points. Thus, for $\phi = 0$ we find $r = a$ as intercept on the polar axis. As ϕ increases from the value 0, r continually increases, reaching the value $2a$ for $\phi = \frac{1}{2}\pi$, and becoming infinite as ϕ approaches the value π.

For any negative value of ϕ (between 0 and $-\pi$) the radius vector has the same length as for the corresponding positive value of ϕ; this means that the parabola is symmetric with respect to the polar axis.

The intersection A of the curve with its axis of symmetry is called the ***vertex***, and the axis of symmetry FA the ***axis***, of the parabola. The segment BB' cut off by the parabola on the perpendicular to the axis drawn through the focus is called the ***latus rectum***; its length is $4a$, if $2a$ is the distance between focus and directrix. Notice also that the vertex A bisects this distance FD so that the distance between focus and vertex as well as that between vertex and directrix is a.

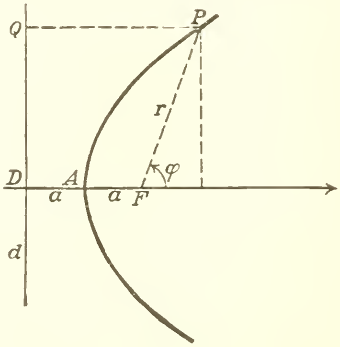


Fig. 64

In Fig. 63 the polar axis is taken positive in the sense from the pole toward the directrix. If the sense from the directrix to the pole is taken as positive (Fig. 64), we have again with F as pole $FP = r$, but the distance of P from the directrix is $2a + r\cos\phi$, so that the polar equation is now

$$r = \frac{2a}{1 - \cos\phi}. \tag{2}$$

We have assumed a as a positive number, $2a$ denoting the absolute value of the distance between the fixed point (focus) and the fixed line (directrix). The radius vector r is then always positive. But the equations (1) and (2) still represent parabolas if a is a negative number, viz. (1) the parabola of Fig. 64, (2) the parabola of Fig. 63, the radius vector r being negative (§ 16).

174. Mechanical Construction. A mechanism for tracing an arc of a parabola consists of a right-angled triangle (shaded in Fig. 65), one of whose sides is applied to the directrix. At a point R of the other side RQ a string of length RQ is attached; the other end of the string is attached at the focus F. As the triangle slides along the directrix, the string is kept taut by means of a pencil at P which traces the parabola. Of course, only a portion of the parabola can thus be traced, since the curve extends to infinity.

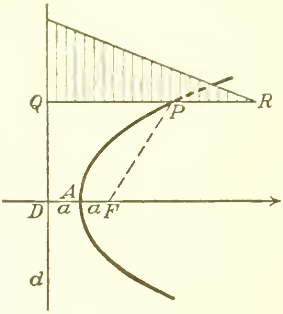


FIG. 65

175. Transformation to Cartesian Coordinates. To obtain the cartesian equation of the parabola let the origin O be taken at the vertex, *i.e.* midway between the fixed line and fixed point, and the axis Ox along the axis of the parabola, positive in the sense from vertex to focus (Fig. 66). Then the focus F has the coordinates $a, 0$, and the equation of the directrix is $x = -a$. The distance FP of any point $P(x, y)$ of the parabola from the focus is therefore $\sqrt{(x-a)^2+y^2}$, and the distance QP of P from the directrix is $a + x$. Hence the equation is

$$(x-a)^2 + y^2 = (a+x)^2,$$

which reduces at once to

$$y^2 = 4ax. \tag{3}$$

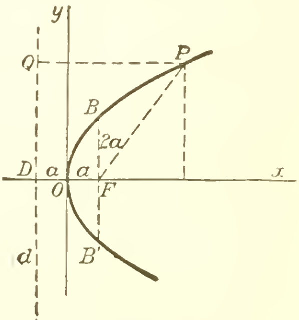


FIG. 66

This then is the ***cartesian equation*** of the parabola, *referred to vertex and axis*, *i.e.* when the vertex is taken as origin and the axis of the parabola (from vertex toward focus) as axis Ox.

Notice that the ordinate at the focus $(a, 0)$ is of length $2a$; the double ordinate $B'B$ at the focus is the latus rectum (§ 173).

176. Negative Values of *a*. In the last article the constant a was again regarded as positive; but (compare § 173) the equation (3) still represents a parabola when a is a negative number, the only difference being that in this case the parabola turns its opening in the negative sense of the axis Ox (toward the left in Fig. 66). Thus the parabolas $y^2 = 4\,ax$ and $y^2 = -4\,ax$ are symmetric to each other with respect to the axis Oy (Ex. 14, p. 138).

The equation (3) is very convenient for plotting a parabola by points. Sketch, with respect to the same axes, the parabolas: $y^2 = 16\,x$, $y^2 = -16\,x$, $y^2 = x$, $y^2 = -x$, $y^2 = 3\,x$, $y^2 = -\frac{1}{5}\,x$.

177. Axis Vertical. The equation

$$x^2 = 4\,ay, \tag{4}$$

which differs from (3) merely by the interchange of x and y, evidently represents a parabola whose vertex lies at the origin and whose axis coincides with the axis Oy. The parabolas (3) and (4) are each the reflection of the other in the line $y = x$ (Ex. 14, p. 138). The equation (4) can be written in the form

$$y = \frac{1}{4\,a}\,x^2.$$

As $1/4\,a$ may be any constant, this is the equation discussed in § 132.

178. New Origin. An equation of the form (Fig. 67)

$$(y - k)^2 = 4\,a(x - h), \tag{5}$$

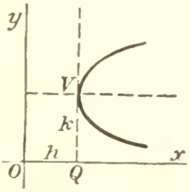


FIG. 67

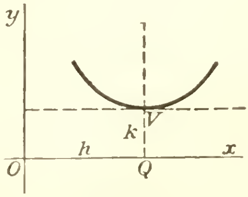


FIG. 68

or of the form (Fig. 68)

$$(x - h)^2 = 4\,a(y - k), \tag{6}$$

evidently represents a parabola whose vertex is the point (h, k), while the axis is in the former case parallel to Ox, in the latter to Oy. For, by taking the point (h, k) as new origin we can reduce these equations to the forms (3), (4), respectively.

The parabola (5) turns its opening to the right or left, the parabola (6) upward or downward, according as $4a$ is positive or negative.

179. General Equation. The equations (5), (6) as well as the equations (3), (4) are of the second degree. Now the general equation of the second degree (§ 79),

$$Ax^2 + 2Hxy + By^2 + 2Gx + 2Fy + C = 0,$$

can be reduced to one of the forms (5), (6) if it contains no term in xy and only one of the terms in x^2 and y^2, *i.e.* if $H = 0$ and either A or B is $= 0$. This reduction is performed (as in § 80) by completing the square in y or x according as the equation contains the term in y^2 or x^2.

Thus *any equation of the second degree, containing no term in xy and only one of the squares x^2, y^2, represents a parabola,* whose vertex is found by completing the square and whose axis is parallel to one of the axes of coordinates.

EXERCISES

1. Sketch the following parabolas:

(*a*) $r = \dfrac{2}{1 + \cos\phi}$. (*b*) $r = \dfrac{10}{1 - \cos\phi}$. (*c*) $r = a\sec^2 \frac{1}{2}\phi$.

2. Sketch the following curves and find their intersections:

(*a*) $r = 8\cos\phi$, $r = \dfrac{2}{1 - \cos\phi}$. (*b*) $r = a$, $r = \dfrac{a}{1 + \cos\phi}$.

(*c*) $r = 4\cos\phi$, $r = \dfrac{8}{1 + \cos\phi}$. (*d*) $r\cos\phi = 2a$, $r = \dfrac{2a}{1 - \cos\phi}$.

3. Sketch the following parabolas:

(*a*) $(y - 2)^2 = 8(x - 5)$. (*b*) $(x + 3)^2 = 5(3 - y)$.

(*c*) $x^2 = 6(y + 1)$. (*d*) $(y + 3)^2 = -3x$.

4. Sketch each of the following parabolas and find the coordinates of the vertex and focus, and the equations of the directrix and axis:

(*a*) $y^2 - 2y - 3x - 2 = 0$. (*b*) $x^2 + 4x - 4y = 0$.
(*c*) $x^2 - 4x + 3y + 1 = 0$. (*d*) $3x^2 - 6x - y = 0$.
(*e*) $8y^2 - 16y + x + 6 = 0$. (*f*) $y^2 + y + x = 0$.
(*g*) $x^2 - x - 3y + 4 = 0$. (*h*) $8y^2 - 3x + 3 = 0$.

5. Sketch the following loci and find their intersections:

(*a*) $y = 2x,\ y = x^2$. (*b*) $y^2 = 4ax,\ x + y = 3a$.
(*c*) $y^2 = x + 3,\ \ y^2 = 5 - x$. (*d*) $y^2 + 4x + 4 = 0,\ x^2 + y^2 = 41$.

6. Sketch the parabolas with the following lines and points as directrices and foci, and find their equations:

(*a*) $x - 4 = 0,\ (6,\ -2)$. (*b*) $y + 3 = 0,\ (0,\ 0)$.
(*c*) $2x + 5 = 0,\ (0,\ -1)$. (*d*) $x = 0,\ (2,\ -3)$.
(*e*) $3y - 1 = 0,\ (-2,\ 1)$. (*f*) $x - 2a = 0,\ (a,\ b)$.

7. Find the parabola, with axis parallel to Ox, and passing through the points:

(*a*) $(1, 0), (5, 4), (10, -6)$. (*b*) $(\frac{10}{3}, -5), (\frac{4}{3}, 0), (\frac{2}{3}, -3)$.
(*c*) $(-1, 5), (3, 1), (\frac{11}{4}, 0)$.

8. Find the parabola, with axis parallel to Oy, and passing through the points:

(*a*) $(0, 0), (-2, 1), (6, 9)$. (*b*) $(1, 4), (4, -1), (-3, 20)$.
(*c*) $(-2, 1), (2, -7), (-3, -2)$.

9. Find the parabola whose directrix is the line $3x - 4y - 10 = 0$ and whose focus is: (*a*) at the origin; (*b*) at $(5, -2)$. Sketch each of these parabolas. When does the equation of a parabola contain an xy term?

10. Find the parabolas with the following points as vertices and foci (two solutions):

(*a*) $(-3, 2), (-3, 5)$. (*b*) $(2, 5), (-1, 5)$.
(*c*) $(-1, -1), (1, -1)$. (*d*) $(0, 0), (0, -a)$.

11. Show that the area of a triangle whose vertices $P_1(x_1,\ y_1)$, $P_2(x_2,\ y_2)$, $P_3(x_3,\ y_3)$ are on the parabola $y^2 = 4ax$, may be expressed by the determinant

$$\frac{1}{8a}\begin{vmatrix} y_1^2 & y_1 & 1 \\ y_2^2 & y_2 & 1 \\ y_3^2 & y_3 & 1 \end{vmatrix} = \frac{1}{8a}(y_2 - y_3)(y_3 - y_1)(y_2 - y_1).$$

12. The area A of a cross-section of a sphere of radius R, at a distance h from the surface, is given by the formula

$$A = 2\,Rh - h^2, \qquad h \leqq R.$$

Reduce this equation to standard form $\bar{A} = k\bar{h}^2$, where $\bar{A}$ and $\bar{h}$ differ from A and h by constants. What is the meaning of $\bar{A}$ and $\bar{h}$?

13. Show that if the area A of the cross-section of any solid perpendicular to a line l, at a distance h from any fixed point P in l, is a quadratic function of h:

$$A = ah^2 + bh + c\,;$$

another point Q in l exists, such that

$$\bar{A} = k\bar{h}^2,$$

where $\bar{h}$ denotes the distance from Q and $\bar{A}$ differs from A by a constant.

14. If s denotes the distance (in feet) from a point P in the line of motion of a falling body, at a time t (in seconds),

$$s - s_0 = \tfrac{1}{2}\,g\,(t - t_0)^2,$$

where g is the gravitational constant (32.2 approximately) and s_0 is the distance from P at the time t_0, show that this equation can be put in the standard form

$$\bar{s} = \tfrac{1}{2}\,g\bar{t}^2,$$

where $\bar{s}$ denotes the distance from some other fixed point in the line of motion and $\bar{t}$ is the time since the body was at that point.

15. The melting point t (in degrees Centigrade) of an alloy of lead and zinc is found to be

$$t = 133 + .875\,x + .01125\,x^2,$$

where x is the percentage of lead in the alloy. Reduce the equation to standard form $\bar{t} = k\bar{x}^2$; and show that $\bar{x} = x - h$, $\bar{t} = t - k$, where h is the percentage of lead that gives the lowest melting point, and k is the temperature at which that alloy melts.

16. Show that the locus of the center of the circle which passes through a fixed point and is tangent to a fixed line is a parabola.

17. Show that the locus of the center of a circle which is tangent to a fixed line and a fixed circle is a parabola. Find the directrix of this parabola.

18. Write in determinant form the equation of the parabola through three given points, $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, $P_3(x_3, y_3)$ with axis parallel to a coordinate axis.

180. Slope of the Parabola. The *slope* $\tan\alpha$ of the parabola

$$y^2 = 4ax$$

at any point $P(x, y)$ (Fig. 69) can be found (comp. § 137) by first determining the slope

$$\tan\alpha_1 = \frac{y_1 - y}{x_1 - x}$$

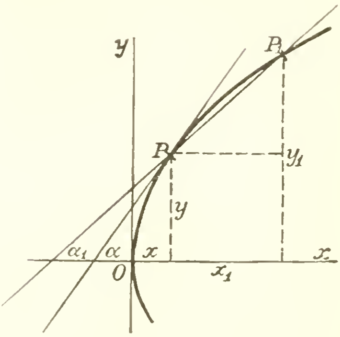


FIG. 69

of the secant PP_1, and then letting $P_1(x_1, y_1)$ move along the curve up to the point $P(x, y)$. Now as P_1 comes to coincide with P, x_1 becomes equal to x, and y_1 equal to y, so that the expression for $\tan\alpha_1$ loses its meaning. But observing that P and P_1 lie on the parabola, we have $y^2 = 4ax$ and $y_1^2 = 4ax_1$, and hence $y_1^2 - y^2 = 4a(x_1 - x)$. Substituting from this relation the value of $x_1 - x$ in the above expression for $\tan\alpha_1$, we find for the slope of the secant:

$$\tan\alpha_1 = 4a\frac{y_1 - y}{y_1^2 - y^2} = \frac{4a}{y_1 + y}.$$

If we now let P_1 come to coincidence with P so that y_1 becomes $= y$, we find for the *slope of the tangent* at $P(x, y)$:

$$(7) \qquad \tan\alpha = \frac{2a}{y}.$$

This slope of the tangent at P is also called the *slope of the parabola* at P. The ordinate y of the parabola is a function of the abscissa x; and the slope of the parabola at $P(x, y)$ is the rate at which y increases with increasing x at P; in other words, it is the *derivative* y' of y with respect to x (compare § 138).

As by the equation of the parabola we have $y = \pm 2\sqrt{ax}$, we find:

$$(8) \qquad y' = \tan \alpha = \frac{2a}{y} = \pm\sqrt{\frac{a}{x}}\cdot$$

The double sign in the last expression corresponds to the fact that to a given value of x belong two points of the curve with equal and opposite slopes.

181. Explicit and Implicit Functions. The result just obtained that when $y^2 = 4ax$ then the derivative of y with respect to x is

$$y' = \frac{2a}{y}$$

can be derived more easily by the general method of the differential calculus. This requires, however, some preliminary explanations.

In the cases in which we have previously determined the derivative y' of a function y of x this function was given ***explicitly***; *i.e.* the equation between x and y that represents the curve was given solved for y, in the form $y = f(x)$.

Our present equation of the parabola, $y^2 = 4ax$, is not solved for y (though it could readily be solved for y by writing it in the form $y = \pm 2\sqrt{ax}$); the same is true of the equation of the circle $x^2 + y^2 = a^2$, or more generally $x^2 + y^2 + ax + by + c = 0$, and also of the general equation of the second degree (§ 79), $Ax^2 + 2Hxy + By^2 + 2Gx + 2Fy + C = 0$. Such equations in x and y, whether they can be solved for y or not, are said to give y ***implicitly*** as a function of x. For, to any particular value of x we can find from such an equation the corresponding values of x (there may be several values; and they may be real or imaginary). Thus, *any equation between x and y, of whatever form, determines y as a function of x.*

182. Derivatives of Implicit Functions. The differential calculus shows that to find the derivative y' of a function y given implicitly by an equation between x and y we have only to differentiate this equation with respect to x, *i.e.* to find the derivative of each term, remembering that y is a function of x. To do this in the simple cases with which we shall have to deal we need only the following two propositions (A) and (B), §§ 183, 184.

183. (*A*) **Derivative of a Function of a Function.** *If u is a function of y, and y a function of x, the derivative of u with respect to x is the product of the derivative of u with respect to y into the derivative y' of y with respect to x.*

For, as u is a function of y which itself is a function of x, u is also a function of x. If x be increased by Δx, y will receive an increment Δy and u an increment Δu. We want to find the derivative of u with respect to x, *i.e.* the limit of $\Delta u/\Delta x$ as Δx approaches zero. Now we can put

$$\frac{\Delta u}{\Delta x}=\frac{\Delta u}{\Delta y}\cdot\frac{\Delta y}{\Delta x};$$

the limit of the first factor, $\Delta u/\Delta y$, is the derivative of u with respect to y, while the limit of the second factor, $\Delta y/\Delta x$, is the derivative y' of y with respect to x.

Thus, if $u = y^n$, we know (§ 151) that the derivative of u *with respect to y* is $= ny^{n-1}$. But if $u = y^n$, and y is a function of x, we can also find the derivative of u *with respect to x*; by the proposition (*A*) it is $ny^{n-1}\cdot y'$. For example, suppose that $u = y^3$, where $y = x^2 - 3x$, so that $u = (x^2 - 3x)^3$. Then the y-derivative of u is $3y^2$; but the x-derivative of u is $3y^2\cdot y' = 3y^2(2x-3) = 3(x^2-3x)^2(2x-3)$. This can readily be verified by expanding $(x^2-3x)^3$ and differentiating the resulting polynomial in the usual way (§ 150).

184. (*B*) **Derivative of a Product.** *If u and v are functions of x, the derivative of uv is u times the derivative of v plus v times the derivative of u:*

$$\text{derivative of } uv = uv' + vu'.$$

For, putting $uv = y$, we have to find the limit of $\Delta y/\Delta x$. When x is increased by Δx, u receives an increment Δu, v an increment Δv, and the increment Δy of y is therefore

$$\Delta y = (u+\Delta u)(v+\Delta v) - uv;$$

dividing by Δx, we find

$$\frac{\Delta y}{\Delta x}=\frac{(u+\Delta u)(v+\Delta v)-uv}{\Delta x}=u\frac{\Delta v}{\Delta x}+v\frac{\Delta u}{\Delta x}+\frac{\Delta u}{\Delta x}\Delta v.$$

In the limit, $\Delta y/\Delta x$ becomes y', $\Delta v/\Delta x$ becomes v'; $\Delta u/\Delta x$ becomes u', and the last term vanishes because its factor Δv becomes zero. Hence:

$$y' = uv' + vu'.$$

185. Computation of Derivatives of Implicit Functions. We are now prepared to find the derivative of y when y is given implicitly as a function of x by the equation $y^2 = 4\,ax$. We have only to differentiate this equation with respect to x, *i.e.* find the x-derivative of each term, *remembering that y is a function of x.* The term y^2, as a function of a function, gives $2\,y \cdot y'$; the term $4\,ax$ gives $4\,a$; hence we find

$$2\,yy' = 4\,a, \quad \text{whence} \quad y' = \frac{2\,a}{y},$$

as in § 180.

Similarly, we find by differentiating the equation of the circle

$$x^2 + y^2 = a^2$$

that
$$2\,x + 2\,yy' = 0,$$

whence
$$y' = -\frac{x}{y};$$

i.e. the slope of the circle $x^2 + y^2 = a^2$ at any point $P(x, y)$ is minus the reciprocal of the slope of the radius through P.

If y is given implicitly as a function of x by the equation

$$x^2 + 5\,xy = 12,$$

which, as we shall see later, represents a hyperbola, we find the derivative of y, *i.e.* the slope of the hyperbola, by differentiating the equation and applying to the second term the proposition (B):

$$2\,x + 5\,x \cdot y' + y \cdot 5 = 0,$$

whence
$$y' = -\frac{5\,y + 2\,x}{5\,x} = -\frac{y}{x} - \frac{2}{5}.$$

EXERCISES

1. Find the derivative of u with respect to x for the following functions:

(*a*) $u = y^2$, when $y = 3\,x - 5$. (*b*) $u = y^3 + 4\,y$, when $y = x^2 - 2\,x$.

(*c*) $u = 2\,y^3 - 3\,y^2$, when $y = x^2 + x$. (*d*) $u = \frac{1}{2}\,y^2 - y$, when $y = x^3$.

2. Find the slope of the following parabolas at the point $P(x, y)$:

(*a*) $y^2 = 5\,x$. (*b*) $y^2 - 5\,y + 6\,x + 4 = 0$. (*c*) $3\,y^2 = 4\,x - 5$

3. Find y' for the following products:

(*a*) $y = x^2(x^3 + 5\,x)$. (*b*) $y = (x + 3)(x - 5)$.

(*c*) $y = (x - a)(x - b)(x - c)$. (*d*) $y = (x - 3)(2\,x + 1)$.

4. Find the slope at the point $P(x, y)$ for each of the following circles by differentiation; compare the results with §§ 88, 89:

(*a*) $x^2 + y^2 = 12$. (*b*) $x^2 + y^2 + ax + by + c = 0$.

(*c*) $Ax^2 + Ay^2 + 2\,Gx + 2\,Fy + C = 0$.

5. Find the slope y' for each of the following curves at the point $P(x, y)$:

(*a*) $xy = a^2$. (*b*) $x^2y - 6\,x + 4 = 0$.

(*c*) $Ax^2 + 2\,Hxy + By^2 + 2\,Gx + 2\,Fy + C = 0$.

186. Equation of the Tangent. As the slope of the parabola

$$y^2 = 4\,ax$$

at the point $P(x, y)$ is $2\,a/y$ (§§ 180–185), ***the equation of the tangent*** at this point is

$$Y - y = \frac{2\,a}{y}(X - x),$$

where X, Y are the coordinates of any point of the tangent, while x, y are the coordinates of the point of contact. This equation can be simplified by multiplying both sides by y and observing that $y^2 = 4\,ax$; we thus find

$$yY = 2\,a\,(x + X). \tag{9}$$

Notice that (as in the case of the circle, § 89) the equation of the tangent is obtained from the equation of the curve, $y^2 = 4\,ax$, by replacing y^2 by yY, $2\,x$ by $x + X$.

The segment TP (Fig. 70) of the tangent from its intersection T with the axis of the parabola to the point of contact P is called the *length of the tangent at P*; the projection TQ of this segment TP on the axis of the parabola is called the ***subtangent*** at P. Now, with $Y = 0$, equation (9) gives $X = -x$, *i.e.* $TO = OQ$; hence *the subtangent is bisected by the vertex.* This furnishes a simple

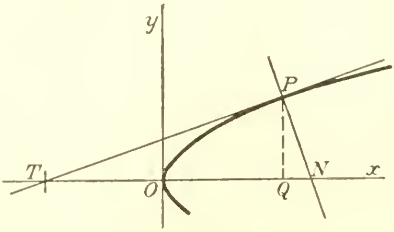


FIG. 70

construction for the tangent at any *point* P of the parabola if the axis and vertex of the parabola are known.

187. Equation of the Normal. The ***normal*** at a point P of any plane curve is defined as the perpendicular to the tangent through the point of contact.

The slope of the normal is therefore (§ 27) minus the reciprocal of that of the tangent. Hence the ***equation of the normal*** to the parabola is:

$$Y - y = -\frac{y}{2a}(X - x),$$

that is:

$$yX + 2aY = (2a + x)y. \tag{10}$$

The segment PN of the normal from the point $P(x, y)$ on the curve to the intersection N of the normal with the axis of the parabola is called the *length of the normal* at P; the projection QN of this segment PN on the axis of the parabola is called the ***subnormal*** at P.

Now, with $Y = 0$, equation (10) gives $X = 2a + x$, and as $x = OQ$, it follows that $QN = 2a$; *i.e. the subnormal of the parabola is constant,* viz. equal to half the latus rectum.

188. Intersections of a Line and a Parabola. The intersections of the parabola

$$y^2 = 4ax$$

with the straight line

$$y = mx + b$$

are found by substituting the value of y from the latter in the former equation:

$$(mx + b)^2 = 4ax,$$

or, reducing:

$$m^2x^2 + 2(mb - 2a)x + b^2 = 0.$$

The roots of this quadratic in x are the abscissas of the points of intersection; the ordinates are then found from

$$y = mx + b.$$

It thus appears that *a straight line cannot intersect a parabola in more than two points.* If the roots are imaginary, the line does not meet the parabola; if they are real and equal, the line has but one point in common with the parabola and is *a tangent to the parabola* (provided $m \neq 0$).

189. Slope Equation of the Tangent. The condition for equal roots is

$$(bm - 2a)^2 = b^2m^2,$$

which reduces to

$$m = \frac{a}{b}.$$

The point that the line of this slope has in common with the parabola is then found to have the coordinates

$$x = \frac{2a - bm}{m^2} = \frac{b^2}{a},\ y = mx + b = 2b.$$

As the slope of the parabola at any point (x, y) is (§ 180) $y' = 2a/y$, the slope at the point just found is $y' = a/b = m$; *i.e.* the slope of the parabola is the same as that of the line $y = mx + b$; this line is therefore a tangent. Thus, *the line*

$$y = mx + \frac{a}{m} \tag{11}$$

is tangent to the parabola $y^2 = 4ax$ whatever the value of m.

This may be called the *slope-form of the equation of the tangent.* Equation (11) can also be deduced from the equation (9), by putting $2a/y = m$ and observing that $y^2 = 4ax$.

190. Slope Equation of the Normal. The equation (10) of the normal can be written in the form

$$Y = -\frac{y}{2a}X + y + \frac{xy}{2a},$$

or since by the equation (3) of the parabola $x = y^2/4a$:

$$Y = -\frac{y}{2a}X + y + \frac{y^3}{8a^2}.$$

If we denote by n the slope of this normal, we have:

$$n = -\frac{y}{2a}, \quad y = -2an, \quad \frac{y^3}{8a^2} = -an^3,$$

so that the equation of the normal assumes the form

$$Y = nX - 2an - an^3. \tag{12}$$

This may be called the *slope-form of the equation of the normal.*

191. Tangents from an Exterior Point. The slope-form (11) of the tangent shows that *from any point* (x, y) *of the plane not more than two tangents can be drawn to the parabola* $y^2 = 4ax$. For, the slopes of these tangents are found by substituting in (11) for x, y the coordinates of the given point and solving the resulting quadratic in m. This quadratic may have real and different, real and equal, or complex roots.

Those points of the plane for which the roots are real and different are said to lie *outside* the parabola; those points for which the roots are imaginary are said to lie *within* the parabola; those points for which the roots are equal lie *on* the parabola.

The quadratic in m can be written

$$xm^2 - ym + a = 0,$$

so that the discriminant is $y^2 - 4ax$. Therefore *a point* (x, y) *of the plane lies within, on, or outside the parabola according as* $y^2 - 4ax$ *is less than, equal to, or greater than zero.*

Similarly, the slope-form (12) of the normal shows that *not more than three normals can be drawn from any point of the plane to the parabola,* since the equation (12) is a cubic for n when the coordinates of any point of the plane are substituted for X, Y. As a cubic has always at least one real root (§ 160), there always exists one normal through a given point; but there may be two or three.

192. Geometric Properties. Let the tangent and normal at P (Fig. 71) meet the axis at T, N; let Q be the foot of the perpendicular from P to the axis, D that of the perpendicular to the directrix d; and let O be the vertex, F the focus.

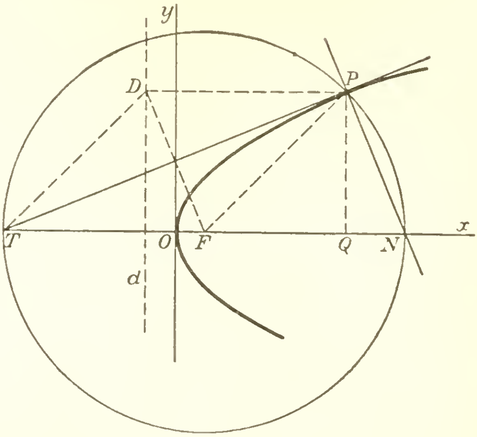


FIG. 71

As the subtangent TQ is bisected by O (§ 186) and the subnormal QN is equal to $2a$ (§ 187), while $OF = a$, it follows that F lies midway between T and N.

The triangle TPN being right-angled at P and F being the midpoint of its hypotenuse, it follows that

$$FP = FT = FN.$$

Hence, if axis and focus are given, the tangent and the normal at any point P of the parabola are found by describing about F a circle through P which will meet the axis at T and N.

As $FP = DP$, it follows that $FPDT$ is a rhombus; the diagonals PT and FD bisect therefore the angles of the rhombus and intersect at right angles. As TP (like TQ) is bisected by the tangent at the vertex, the intersection of these diagonals lies on this tangent at the vertex. The properties just proved that *the tangent at P bisects the angle between the focal radius PF and the parallel PD to the axis* and that *the perpendicular from the focus to the tangent meets the tangent on the tangent at the vertex* are of particular importance.

193. Diameters. It is known from elementary geometry that in a circle all chords parallel to any given direction have their midpoints on a straight line which is a diameter of the circle.

Similarly, in a parabola, *the locus of the midpoints of all chords parallel to any given direction is a straight line*, and this line which is parallel to the axis is called a **diameter** of the parabola. To prove this, take the vertex as origin and the axis of the parabola as axis Ox (Fig. 72) so that the equation is $y^2 = 4ax$. Any line of given slope m has the equation

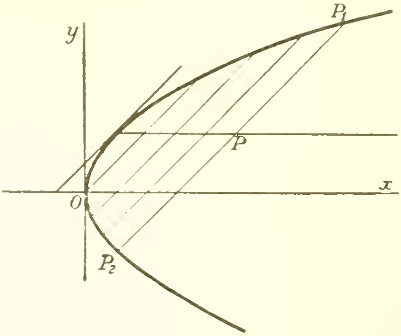


FIG. 72

$$y = mx + b,$$

and with variable b this represents a pencil of parallel lines. Eliminating x we find for y the quadratic

$$y^2 - \frac{4a}{m}y + \frac{4ab}{m} = 0.$$

The roots y_1, y_2 are the ordinates of the points P_1, P_2 at which the line intersects the parabola. The sum of the roots is

$$y_1 + y_2 = \frac{4a}{m};$$

hence the ordinate $\frac{1}{2}(y_1 + y_2)$ of the midpoint P between P_1, P_2 is constant (*i.e.* independent of x), viz. $= 2a/m$, and independent of b. The midpoints of all chords of the same slope m lie, therefore, on a parallel to the axis, at the distance $2a/m$ from it.

The condition for equal roots (§ 189) gives $b = a/m$. That one of the parallels which passes through the point where the diameter meets the parabola is, therefore,

$$y = mx + \frac{a}{m};$$

by § 189 this is a tangent. Thus, *the tangent at the end of a diameter is parallel to the chords bisected by the diameter.*

EXERCISES

1. Find and sketch the tangent and normal of the following parabolas at the given points:

(*a*) $2y^2 = 25x$, $(2, 5)$. (*b*) $3y^2 = 4x$, $(3, -2)$. (*c*) $y^2 = 2x$, $(\frac{1}{2}, 1)$.
(*d*) $5y^2 = 12x$, $(\frac{5}{3}, -2)$. (*e*) $y^2 = x$, $(1, 1)$. (*f*) $45y^2 = x$, $(5, \frac{1}{3})$.

2. Show that the secant through the points $P(x, y)$ and $P_1(x_1, y_1)$ of the parabola $y^2 = 4ax$ has the equation $4aX - (y + y_1)Y + yy_1 = 0$, and that this reduces to the tangent at P when P_1 and P coincide.

3. Find the angle between the tangents to a parabola at the vertex and at the end of the latus rectum. Show that the tangents at the ends of the latus rectum are at right angles.

4. Find the length of the tangent, subtangent, normal, and subnormal of the parabola $y^2 = 4x$ at the point $(1, 2)$.

5. Find and sketch the tangents to the parabola $y^2 = 8x$ from each of the following points:

(*a*) $(-2, 3)$. (*b*) $(-2, 0)$. (*c*) $(-6, 0)$. (*d*) $(8, 8)$.

6. Draw the tangents to the parabola $y^2 = 3x$ that are inclined to the axis Ox at the angles: (*a*) 30°, (*b*) 45°, (*c*) 135°, (*d*) 150°; and find their equations.

7. Find and sketch the tangents to the parabola $y^2 = 4x$ that pass through the point $(-2, 2)$.

8. Find and sketch the normals to the parabola $y^2 = 6x$ that pass through the points:

(*a*) $(\frac{3}{2}, 0)$. (*b*) $(\frac{15}{2}, -3)$. (*c*) $(\frac{45}{8}, -\frac{9}{8})$. (*d*) $(\frac{9}{8}, -\frac{9}{8})$. (*e*) $(0, 0)$.

9. Are the following points inside, outside, or on the parabola $8y^2 = x$? (*a*) $(3, 1)$. (*b*) $(2, \frac{1}{2})$. (*c*) $(8, \frac{7}{8})$. (*d*) $(10, \frac{3}{2})$.

10. Show that any tangent to a parabola intersects the directrix and latus rectum (produced) in points equally distant from the focus.

11. Show that the tangents drawn to a parabola from any point of the directrix are perpendicular.

12. Show that the ordinate of the intersection of any two tangents to the parabola $y^2 = 4ax$ is the arithmetic mean of the ordinates of the points of contact, and the abscissa is the geometric mean of the abscissas of the points of contact.

13. Show that the sum of the slopes of any two tangents of the parabola $y^2 = 4\,ax$ is equal to the slope Y/X of the radius vector of the point of intersection (X, Y) of the tangents; find the product of the slopes.

14. Find the locus of the intersection of two tangents to the parabola $y^2 = 4\,ax$. if the sum of the slopes of the tangents is constant.

15. Find the locus of the intersection of two perpendicular tangents to a parabola; of two perpendicular normals to a parabola.

16. Show that the angle between any two tangents to a parabola is half the angle between the focal radii of the points of contact.

17. From the vertex of a parabola any two perpendicular lines are drawn; show that the line joining their other intersections with the parabola cuts the axis at a fixed point.

18. Find and sketch the diameter of the parabola $y^2 = 6\,x$ that bisects the chords parallel to $3\,x - 2\,y + 5 = 0$; give the equation of the focal chord of this system.

19. Find the system of parallel chords of the parabola $y^2 = 8\,x$ bisected by the line $y = 3$.

20. Find the diameter and corresponding chord of the parabola $y^2 = 4\,x$ that pass through the point $(5, -2)$; at what angle does this diameter meet its chord?

21. Show that the tangents at the extremities of any chord of a parabola intersect on the diameter bisecting this chord. Compare Ex. 12.

22. Find the length of the focal chord of a parabola of given slope m.

23. Find the tangent and normal to the parabola $x^2 = 4\,ay$ in terms of the coordinates of the point of contact.

24. Find the angles at which the parabolas $y^2 = 4\,ax$ and $x^2 = 4\,ay$ intersect.

25. If the vertex of a right angle moves along a fixed line while one side of the angle always passes through a fixed point, the other side *envelopes a parabola* (*i.e.* is always a tangent to the parabola). The fixed line is the tangent at the vertex, the fixed point is the focus of the parabola.

26. Two equal confocal parabolas have the same axis but open in opposite sense; show that they intersect at right angles.

27. If axis, vertex, and one other point of the parabola are given, additional points can be constructed as follows: Let O be the vertex, P the given point, and Q the foot of the perpendicular from P to the tangent at the vertex; divide QP into equal parts by the points $A_1, A_2, \cdots$; and OQ into the same number of equal parts by the points $B_1, B_2, \cdots$; the intersections of OA_1, OA_2, $\cdots$ with the parallels to the axis through B_1, B_2, $\cdots$ are points of the parabola.

28. If two tangents AP_1, AP_2 to a parabola with their points of contact P_1, P_2 are given and AP_1, AP_2 be divided into the same number of equal parts, the points of division being numbered from P_1 to A and from A to P_2, the lines joining the points bearing equal numbers are tangents to the parabola. To prove this show that the intersections of any tangent with the lines AP_1, AP_2 divide the segments P_1A, AP_2 in the same division ratio.

29. The shape assumed by a uniform chain or cable suspended between two fixed points P_1, P_2 is called a *catenary;* its equation is not algebraic and cannot be given here. But when the line P_1P_2 is nearly horizontal and the depth of the lowest point below P_1P_2 is small in comparison with P_1P_2, the catenary agrees very nearly with a parabola.

The distance between two telegraph poles is 120 ft.; P_2 lies 2 ft. above the level of P_1; and the lowest point of the wire is at 1/3 the distance between the poles. Find the equation of the parabola referred to P_1 as origin and the horizontal line through P_1 as axis Ox; determine the position of the lowest point and the ordinates at intervals of 20 ft.

30. The cable of a suspension bridge assumes the shape of a parabola if the weight of the suspended roadbed (together with that of the cables) is uniformly distributed horizontally. Suppose the towers of a bridge 240 ft. long are 60 ft. high and the lowest point of the cables is 20 ft. above the roadway; find the vertical distances from the roadway to the cables at intervals of 20 ft.

31. When a parabola revolves about its axis, it generates a surface called a paraboloid of revolution; all meridian sections (sections through the axis) are equal parabolas. If the mirror of a reflecting telescope is such a surface (the portion about the vertex), all rays of light falling in parallel to the axis are reflected to the same point; explain why.

194. Parameter Equations. Instead of using the cartesian or polar equation of a curve it is often more convenient to express x and y (or r and ϕ) each in terms of a third variable, which is then called the ***parameter.***

Thus the *parameter equations of a circle* of radius a about the origin as center are:

$$x = a \cos \phi, \quad y = a \sin \phi,$$

ϕ being the parameter. To every value of ϕ corresponds a definite x and a definite y, and hence a point of the curve. The elimination of ϕ, by squaring and adding the equations, gives the cartesian equation $x^2 + y^2 = a^2$.

Again, to determine the *motion of a projectile* we may observe that, if gravity were not acting, the projectile, started with an initial velocity v_0 at an angle ϵ to the horizon would have at the time t the position

$$x = v_0 \cos \epsilon \cdot t, \quad y = v_0 \sin \epsilon \cdot t,$$

the horizontal as well as the vertical motion being uniform. But, owing to the constant acceleration g of gravity (downward), the ordinate y is diminished by $\frac{1}{2} gt^2$ in the time t, so that the coordinates of the projectile at the time t are

$$x = v_0 \cos \epsilon \cdot t, \quad y = v_0 \sin \epsilon \cdot t - \tfrac{1}{2} gt^2.$$

These are the parameter equations of the path, the parameter here being the time t. The elimination of t gives the cartesian equation of the parabola described by the projectile:

$$y = v_0 \tan \epsilon \cdot x - \frac{g}{2 v_0^2 \cos^2 \epsilon} x^2.$$

195. Parameter Equations of a Parabola. For any parabola $y^2 = 4\, ax$ we can also use as parameter the angle α made by the tangent with the axis Ox; we have for this angle (§ 180):

$$\tan \alpha = \frac{2\, a}{y};$$

it follows that $y = 2\, a \operatorname{ctn} \alpha$ and hence $x = y^2/4\, a = a \operatorname{ctn}^2 \alpha$.

The equations

$$x = a \operatorname{ctn}^2 \alpha, \quad y = 2\, a \operatorname{ctn} \alpha$$

are *paramenter equations of the parabola* $y^2 = 4\, ax$; the elimination of $\cot \alpha$ gives the cartesian equation.

196. Parabola referred to Diameter and Tangent. The equation of the parabola $y^2 = 4\, ax$ preserves this simple form if instead of axis and tangent at the vertex we take as axes any diameter and the tangent at its end. The equation in these oblique coordinates is

$$y_1^2 = 4\, a_1 x_1,$$

where $a_1 = a/\sin^2 \alpha$, α being the angle between the axes, *i.e.* the slope angle at the new origin O_1 (Fig. 73).

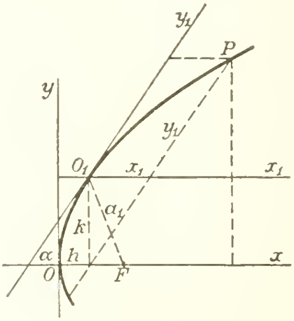


FIG. 73

To prove this observe that as the new origin $O_1\,(h, k)$ is a point of the parabola $y^2 = 4\, ax$ we have by § 195

$$h = a \operatorname{ctn}^2 \alpha, \quad k = 2\, a \operatorname{ctn} \alpha,$$

α being the angle at which the tangent at O_1 is inclined to the axis. Hence, transferring to parallel axes through O_1, we obtain the equation

$$(y + 2\, a \operatorname{ctn} \alpha)^2 = 4\, a\,(x + a \operatorname{ctn}^2 \alpha),$$

which reduces to

$$y^2 + 4\, a \operatorname{ctn} \alpha \cdot y = 4\, ax.$$

The relation between the rectangular coordinates x, y and the oblique coordinates x_1, y_1, both with O_1 as origin, is seen from the figure to be

$$x = x_1 + y_1 \cos \alpha, \quad y = y_1 \sin \alpha.$$

Substituting these values we find

$$y_1^2 \sin^2 \alpha + 4\, a \cos \alpha \cdot y_1 = 4\, ax_1 + 4\, ay_1 \cos \alpha,$$

i.e.

$$y_1^2 = 4 \frac{a}{\sin^2 \alpha} x_1 = 4\, a_1 x_1,$$

if we put $a/\sin^2 \alpha = a_1$.

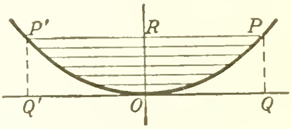
The meaning of the constant a_1 appears by observing that

$$a_1 = \frac{a}{\sin^2 \alpha} = \frac{1 + \tan^2 \alpha}{\tan^2 \alpha} a = a \operatorname{ctn}^2 \alpha + a = h + a;$$

a_1 is therefore the distance of the new origin O_1 from the directrix, or what amounts to the same, from the focus F.

197. Area of Parabolic Segment.

A parabola, together with any chord perpendicular to its axis, bounds an area OPP' (shaded in Fig. 74). It was shown by Archimedes (about 250 B.C.) that this area is two thirds the area of the rectangle $PP'Q'Q$ that has the chord $P'P$ as one side and the tangent at the vertex as opposite side.



'FIG. 74

This rectangle $PP'Q'Q$ is often called (somewhat improperly) the circumscribed rectangle so that the result can be expressed briefly by saying that *the area of the parabola is 2/3 of that of the circumscribed rectangle.*

This statement is of course equivalent to saying that *the* (non-shaded) *area* OQP *is* 1/3 *of the area of the rectangle* $OQPR$. In this form the proposition is proved in the next article.

198. Area by Approximation Process.

To obtain first an *approximate* value (A) for the area OQP (Fig. 75) we may subdivide the area into rectangular strips of equal width, by dividing OQ into, say, n equal parts and drawing the ordinates $y_1, y_2, \cdots y_n$. If the width of these strips is Δx so that $OQ = n\Delta x$, we have as approximate value of the area:

$$(A) = \Delta x \cdot y_1 + \Delta x \cdot y_2 + \cdots + \Delta x \cdot y_n.$$

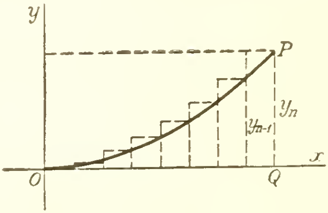


FIG. 75

Now y_1 is the ordinate corresponding to the abscissa Δx; y_2 corresponds to the abscissa $2\,\Delta x$, etc.; y_n corresponds to the abscissa $n\Delta x = OQ$. Hence, if the equation of the curve is $x^2 = 4\,ay$, we have:

$$y_1 = \frac{1}{4\,a}(\Delta x)^2, \quad y_2 = \frac{1}{4\,a}(2\,\Delta x)^2, \quad \cdots\ y_n = \frac{1}{4\,a}(n\Delta x)^2.$$

Substituting these values we find:

$$(A) = \frac{(\Delta x)^3}{4\,a}(1 + 2^2 + 3^2 + \cdots + n^2).$$

By Ex. 37, p. 74,

$$1^2+2^2+\cdots+n^2 = \tfrac16 n(n+1)(2n+1) = \tfrac16(2n^3+3n^2+n);$$

hence
$$A_n = \frac{\Delta x^3}{24a}(2n^3+3n^2+n)$$
$$= \frac{n^3\Delta x^3}{24a}\left(2+\frac3n+\frac1{n^2}\right).$$

Now $n\Delta x = OQ = x_n$, the abscissa of the terminal point P, whatever the number n and length Δx of the subdivisions. Hence, if we let the number n increase indefinitely, we find in the limit the *exact* expression A for the area OQP:

$$A = \frac{x_n^3}{12a} = \frac13 x_n\cdot\frac{x_n^2}{4a} = \frac13 x_n y_n,$$

where $y_n = x_n^2/4a$ is the ordinate of the terminal point P. As $x_n y_n$ is the area of the rectangle $OQPR$, our proposition is proved.

The integral calculus furnishes a far more simple and more general method for finding the area under a curve. The method used above happens to succeed in the simple case of the parabola because we can express the sum $1^2+2^2+3^2+\cdots+n^2$ in a simple form.

199. Area expressed in Terms of Ordinates. The area (shaded in Fig. 76) between the parabola $x^2 = 4ay$, the axis Ox, and the two ordinates y_1, y_3, whose abscissas differ by $2\Delta x$ is evidently, by the formula of § 198,

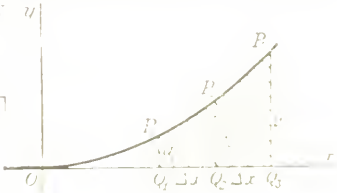


Fig. 76

$$A = \frac1{12a}(x_3^3 - x_1^3) = \frac1{12a}[(x_1+2\Delta x)^3 - x_1^3]$$
$$= \frac{\Delta x}{12a}(6x_1^2+12x_1\Delta x+8(\Delta x)^2).$$

This expression can be given a remarkably simple form by introducing not only the ordinates $y_1 = x_1^2/4a$, $y_3 = (x_1+2\Delta x)^2/4a$, but also the ordinate y_2 midway between y_1 and y_3, whose abscissa is $x_1+\Delta x$. For we have:

$$y_1+4y_2+y_3 = \frac1{4a}[x_1^2+4(x_1+\Delta x)^2+(x_1+2\Delta x)^2]$$
$$= \frac1{4a}(6x_1^2+12x_1\Delta x+8(\Delta x)^2).$$

We find therefore:

$$A = \tfrac{1}{3}\,\Delta x(y_1 + 4\,y_2 + y_3).$$

This formula holds not only when the vertex of the parabola is at the origin, but also when it is at any point (h, k), provided the axis of the parabola is parallel to Oy.

For (Fig. 77), to find the area under the arc $P_1P_2P_3$ we have only to add to the doubly shaded area the simply shaded rectangle whose area is $2\,k\Delta x$. We find therefore for the whole area:

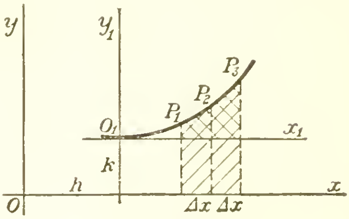


FIG. 77

$$\begin{aligned}\tfrac{1}{3}\,\Delta x(y_1 + 4\,y_2 + y_3) + 2\,k\Delta x &= \tfrac{1}{3}\,\Delta x(y_1 + 4\,y_2 + y_3 + 6\,k)\\ &= \tfrac{1}{3}\,\Delta x\,[(y_1 + k) + 4\,(y_2 + k) + (y_3 + k)],\end{aligned}$$

where y_1, y_2, y_3 are the ordinates of the parabola referred to its vertex, and hence $y_1 + k$, $y_2 + k$, $y_3 + k$ the ordinates for the origin O.

We have therefore for any parabola whose axis is parallel to Oy:

$$A = \tfrac{1}{3}\,\Delta x(y_1 + 4\,y_2 + y_3).$$

200. Approximation to any Area. Simpson's Rule. The last formula is sometimes used to find an ***approximate value*** for the ***area under any curve*** (*i.e.* the area bounded by the axis Ox, an arc AB of the curve, and the ordinates of A and B, Fig. 78). This method is particularly convenient if a number of equidistant ordinates of the curve are known, or can be found graphically.

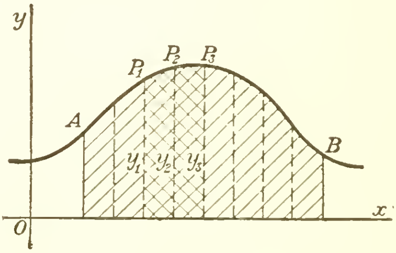


FIG. 78

Let Δx be the distance of the ordinates, and let y_1, y_2, y_3 be any three consecutive ordinates. Then the doubly shaded portion of the required area, between y_1 and y_3, will be (if Δx is sufficiently small) very nearly equal to the area under the parabola that passes through P_1, P_2, P_3 and has its axis parallel to Oy. This parabolic area is by § 199

$$= \tfrac{1}{3}\,\Delta x(y_1 + 4\,y_2 + y_3).$$

The whole area under AB is a sum of such expressions. This method for finding an approximate expression for the area under any curve is known as *Simpson's rule* (Thomas Simpson, 1743) although the fundamental idea of replacing an arc of the curve by a parabolic arc had been suggested previously by Newton.

201. Area of any Parabolic Segment. As the equation of a parabola referred to any diameter and the tangent at its end has exactly the same form as when the parabola is referred to its axis and the tangent at the vertex (§ 196), it can easily be shown that *the area of any parabolic segment is 2/3 of the area of the circumscribed parallelogram.* In this statement the parabolic segment is understood to be bounded by any arc of the parabola and its chord; and the circumscribed parallelogram is meant to have for two of its sides the chord and the parallel tangent while the other two sides are parallels to the axis through the extremities of the chord (Fig. 79).

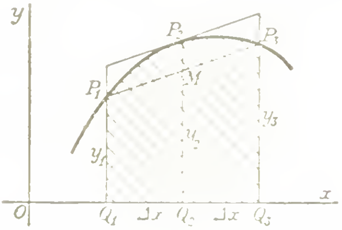


FIG. 79

With the aid of this proposition Simpson's rule can be proved very simply. For, the area of the parabolic segment $P_1P_2P_3$ (Fig. 79) is then equal to 2/3 of the parallelogram formed by the chord P_1P_3, the tangent at P_2, and the ordinates y_1, y_3 (produced if necessary). This parallelogram has a height $= 2\,\Delta x$ and a base $= MP_2 = y_2 - \frac{1}{2}(y_1 + y_3)$; hence the area of $P_1P_2P_3$ is

$$= \tfrac{2}{3}\,\Delta x\,(2\,y_2 - y_1 - y_3) = \tfrac{1}{3}\,\Delta x\,[4\,y_2 - 2\,(y_1 + y_3)].$$

To find the whole shaded area we have only to add to this the area of the trapezoid $Q_1Q_3P_3P_1$ which is

$$= \Delta x\,(y_1 + y_3).$$

$$\text{Hence } A = Q_1Q_3P_3P_2P_1 = \tfrac{1}{3}\,\Delta x\,[4\,y_2 - 2\,(y_1 + y_3) + 3\,(y_1 + y_3)]$$
$$= \tfrac{1}{3}\,\Delta x\,(y_1 + 4\,y_2 + y_3).$$

EXERCISES

1. Show that the area of any parabolic segment is 2/3 of the area of the circumscribed parallelogram.

2. In what ratio does the parabola $y^2 = 4\,ax$ divide the area of the circle $(x - a)^2 + y^2 = 4\,a^2$?

3. Find the area bounded by the parabola $y^2 = 4\,ax$ and a line of slope m through the focus.

4. By a method similar to that used in finding the area of a parabola (§ 198), find exactly the area bounded by the curve $y = x^3$, the axis Ox, and the line $x = a$. What is the area bounded by this same curve, the axis Ox, and the lines $x = a$, $x = b$? What is the area bounded by the curve $y = x^3 + c$, the axis Ox, and the lines $x = a$, $x = b$?

5. Find and sketch the curve whose ordinates represent the area bounded by: (*a*) the line $y = \frac{2}{3}x$, the axis Ox, and any ordinate, (*b*) the parabola $y = \frac{2}{3}x^2$, the axis Ox, and any ordinate.

6. Let $P_1(x_1, y_1)$, $P_2(x_1 + \Delta x, y_2)$, $P_3(x_1 + 2\,\Delta x, y_3)$ be three points of a curve. Let A denote the sum of the areas of the two trapezoids formed by the chords P_1P_2, P_2P_3, the axis Ox, and the ordinates y_1, y_2, y_3. Let B denote the area of the trapezoid formed by any line through P_2, the axis Ox, and the segments cut off on the ordinates y_1, y_3. Find the approximation to the area under the curve given by each of the following formulas: $\frac{1}{2}(A + B)$, $\frac{1}{3}(2\,A + B)$, $\frac{1}{3}(A + 2\,B)$. Which of these gives Simpson's rule?

7. To find an approximation to the area bounded by a curve, the axis Ox, and two ordinates, divide the interval into any even number of strips of equal width and apply Simpson's rule to each successive pair. Show that the result found is: the sum of the extreme ordinates plus twice the sum of the other odd ordinates plus four times the sum of the even ordinates, multiplied by one third the distance between the ordinates.

8. Find an approximation to the areas bounded by the following curves and the axis Ox (divide the interval in each case into eight or more equal parts):

(*a*) $4\,y = 16 - x^2$. (*b*) $y = (x + 3)(x - 2)^2$. (*c*) $y = x^2 - x^3$.

9. The cross-sections in square feet of a log at intervals of 6 ft. are 3.25, 4.27, 5.34, 6.02, 6.83: find the volume.

10. The cross sections of a vessel in square feet measured at intervals of 3 ft. are 0, 2250, 5800, 8000, 10200; find the volume. Allowing one ton for each 35 cu. ft., what is the displacement of the vessel?

11. The half-widths in feet of a launch's deck at intervals of 5 ft. are 0, 1.8, 2.6, 3.2, 3.3, 3.3, 2.7, 2.1, 1; find the area.

202. Shearing Force and Bending Moment. A straight beam AB (Fig. 80), of length l, fixed at one end A in a horizontal position and loaded uniformly with w lb. per unit of length, will bend under the load. At any point P, at the distance x from A, the effect of the load $w(l - x)$ that rests on PB is twofold:

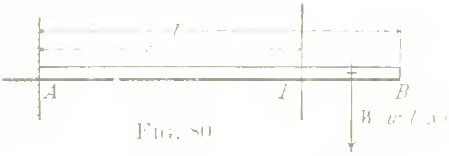


FIG. 80

(a) If the beam were cut at P, this load, which is equivalent to a single force $W = w(l - x)$ applied at the midpoint of PB, would pull the portion PB vertically down. This force which tends to shear off the beam at P is called the *shearing force* F at P. Adopting the convention that downward forces are to be regarded as positive, we have

$$F = w(l - x).$$

The shearing force at the various points of AB is therefore represented by the ordinates of the straight line CB (Fig. 81).

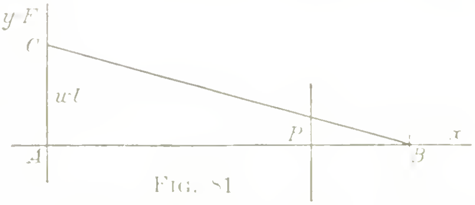


FIG. 81

(b) If the beam were hinged at P, the effect of the load $w(l - x)$ on PB would be to turn it about P. As the force $w(l - x)$ can be regarded as applied at the midpoint of PB, this effect at P is represented by the *bending moment*

$$M = -\tfrac{1}{2}w(l - x)^2,$$

the minus sign arising from the convention of regarding a moment as positive when tending to turn counterclockwise. As $w(l - x)$ turns clockwise about P, the moment is negative. The curve DB representing the bending moments (Fig. 82) is a parabola.

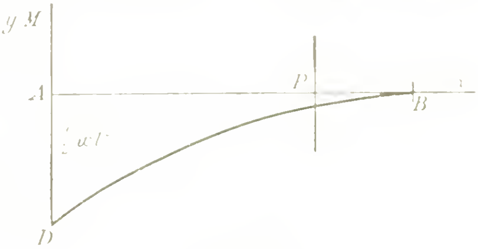


FIG. 82

More briefly we may say that the single force $F = w(l - x)$ applied at the midpoint of PB is equivalent to an equal force at P, the shear $F = w(l - x)$, together with the couple formed by $+F$ at the midpoint of PB and $-F$ at P; the moment of this couple is the bending moment $M = -\frac{1}{2}w(l - x)^2$.

203. Relation of Bending Moment to Shearing Force. For any beam AB, fixed at one or both ends or supported freely at two or more points, in a horizontal position, and loaded by any vertical forces, the *shearing force* at any point P is defined as the algebraic sum of all the forces (including the reactions of the supports) on one side of P, and the *bending moment* at P as the algebraic sum of the moments of these forces about P.

It may be noted that if the shear F is constant, the bending moment is a linear function of x (*i.e.* of the abscissa of P); if F (as in § 202) is a linear function of x, M is a quadratic function; in either case the derivative of M with respect to x is equal to F:

$$M' = F.$$

It follows that the bending moment is a maximum or minimum at any point where the shear is zero.

EXERCISES

Determine F and M as functions of x for a horizontal beam AB of length l and represent F and M graphically:

1. When the beam is fixed at one end A (cantilever) and carries a single load W at the other end B.

2. When the beam is freely supported at its ends A, B and loaded: (*a*) uniformly with w lb. per unit of length; (*b*) with a single load W at the midpoint; (*c*) with a single load W at the distance a from A. Determine first the reactions at A and B.

3. When the beam is supported at the two points trisecting it and carries: (*a*) a uniform load w lb./ft.; (*b*) a single load W at A and at B.

4. When the beam is supported at its ends and is loaded: (*a*) with w lb./ft. over the middle third; (*b*) with w lb./ft. over the first and third thirds; (*c*) with w lb./ft. over the first half and $2w$ lb./ft. over the second half.

5. When the beam is fixed at A and carries w lb./ft. over the outer half.

CHAPTER X

ELLIPSE AND HYPERBOLA

204. Definition of the Ellipse. The ***ellipse*** may be defined as *the locus of a point whose distances from two fixed points have a constant sum.*

If F_1, F_2 (Fig. 83) are the fixed points, which are called the ***foci***, and if P is any point of the ellipse, the condition to be satisfied by P is

$$F_1P + F_2P = 2\,a.$$

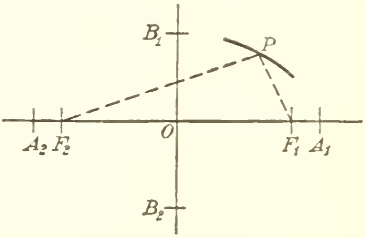


FIG. 83

The ellipse can be traced mechanically by attaching at F_1, F_2 the ends of a string of length $2\,a$ and keeping the string taut by means of a pencil. It is obvious that the curve will be symmetric with respect to the line F_1F_2, and also with respect to the perpendicular bisector of F_1F_2. These axes of symmetry are called the ***axes*** of the ellipse; their intersection O is called the ***center*** of the ellipse.

205. Axes. The points A_1, A_2, B_1, B_2 (Figs. 83 and 84) where the ellipse intersects these axes are called ***vertices.*** The distance A_2A_1 of those vertices that lie on the axis containing the foci F_1, F_2 is $= 2\,a$, the length of the string. For when the point P in describing the ellipse arrives at A_1, the string is doubled along F_1A_1 so that

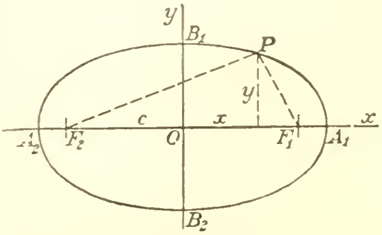


FIG. 84

$$F_2F_1 + 2\,F_1A_1 = 2\,a:$$

and since, by symmetry, $A_2F_2 = F_1A_1$, we have

$$A_2F_2 + F_2F_1 + F_1A_1 = A_2A_1 = 2\,a.$$

The distance $A_2A_1 = 2\,a$, which is called the ***major axis***, must evidently be not less than the distance F_2F_1 between the foci, which we shall denote by $2\,c$.

The distance B_2B_1 of the other two vertices is called the ***minor axis*** and will be denoted by $2\,b$. We then have

$$b^2 = a^2 - c^2;$$

for when P arrives at B_1, we have $B_1F_2 = B_1F_1 = a$.

206. Equation of the Ellipse. If we take the center O as origin and the axis containing the foci as axis Ox, the equation of the ellipse is readily found from the condition $F_1P + F_2P = 2\,a$, which gives, since the coordinates of the foci are c, 0 and $-c$, 0:

$$\sqrt{(x-c)^2+y^2} + \sqrt{(x+c)^2+y^2} = 2\,a.$$

Squaring both members we have

$$x^2+y^2+c^2+\sqrt{(x^2+y^2+c^2-2\,cx)\,(x^2+y^2+c^2+2\,cx)} = 2\,a^2;$$

transferring $x^2+y^2+c^2$ to the right-hand member and squaring again, we find

$$(x^2+y^2+c^2)^2 - 4\,c^2x^2 = 4\,a^4 - 4\,a^2\,(x^2+y^2+c^2) + (x^2+y^2+c^2)^2,$$

i.e.

$$(a^2-c^2)\,x^2 + a^2y^2 = a^2(a^2-c^2).$$

Now for the ellipse (§ 205) $a^2-c^2=b^2$. Hence, dividing both members by a^2b^2, we find

$$(1)\qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

as the ***cartesian equation*** *of the ellipse referred to its axes.*

This equation shows at a glance: (a) that the curve is symmetric to Ox as well as to Oy; (b) that the intercepts on the axes Ox, Oy are $\pm a$, and $\pm b$. The lengths a, b are called the *semi-axes.*

Solving the equation for y we find

$$(2) \qquad y = \pm \frac{b}{a}\sqrt{a^2 - x^2},$$

which shows that the curve does not extend beyond the vertex A_1 on the right, nor beyond A_2 on the left.

If a and b (or, what amounts to the same, a and c) are given numerically, we can calculate from (2) the ordinates of as many points as we please. If, in particular, $a = b$ (and hence $c = 0$) the ellipse reduces to a *circle.*

EXERCISES

1. Sketch the ellipse of semi-axes $a = 4$, $b = 3$, by marking the vertices, constructing the foci, and determining a few points of the curve from the property $F_1P + F_2P = 2a$. Write down the equation of this ellipse, referred to its axes.

2. Sketch the ellipse $x^2/16 + y^2/9 = 1$ by drawing the circumscribed rectangle and finding some points from the equation solved for y.

3. Sketch the ellipses: (*a*) $x^2 + 2y^2 = 1$. (*b*) $3x^2 + 12y^2 = 5$.
(*c*) $3x^2 + 3y^2 = 20$. (*d*) $x^2 + 20y^2 = 1$.

4. If in equation (1) $a < b$, the equation represents an ellipse whose foci lie on Oy. Sketch the ellipses:

(*a*) $\frac{x^2}{4} + \frac{y^2}{16} = 1$. (*b*) $20x^2 + y^2 = 1$. (*c*) $10x^2 + 9y^2 = 10$.

5. Find the equation of the ellipse referred to its axes when the foci are midpoints between the center and vertices.

6. Find the product of the slopes of chords joining any point of an ellipse to the ends of the major axis. What value does this product assume when the ellipse becomes a circle?

7. Derive the equation of the ellipse with foci at $(0, c)$, $(0, -c)$, and major axis $2a$.

8. Write the equations of the following ellipses: (*a*) with vertices at $(5, 0)$, $(-5, 0)$, $(0, 4)$, $(0, -4)$; (*b*) with foci at $(2, 0)$, $(-2, 0)$, and major axis 6.

9. Find the equation of the ellipse with foci at $(1, 1)$, $(-1, -1)$, and major axis 6, and sketch the curve.

207. Definition of the Hyperbola. The ***hyperbola*** can be defined as *the locus of a point whose distances from two fixed points have a constant difference.*

The fixed points F_1, F_2 are again called the ***foci;*** if $2a$ is the constant difference, every point P of the hyperbola must satisfy the condition

$$F_2P - F_1P = \pm 2a.$$

Notice that the length $2a$ must here be not greater than the distance $F_2F_1 = 2c$ of the foci.

The curve is symmetric to the line F_2F_1 and to its perpendicular bisector.

A mechanism for tracing an arc of a hyperbola consists of a straightedge F_2Q (Fig. 85) which turns about one of the foci, F_2; a string, of length $F_2Q - 2a$, is fastened to the straightedge at Q and with its other end to the other focus, F_1. As the straightedge turns about F_2, the string is kept taut by means of a pencil at P which describes the hyperbolic arc. Of course only a portion of the hyperbola can be traced in this manner.

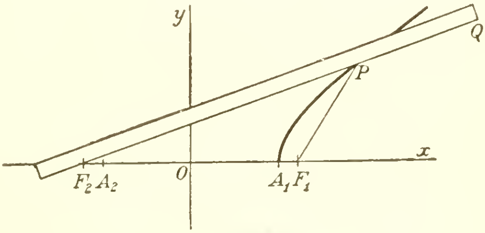


FIG. 85

208. Equation of the Hyperbola. If the line F_2F_1 be taken as the axis Ox, its perpendicular bisector as the axis Oy, and if $F_2F_1 = 2c$, the condition $F_2P - F_1P = \pm 2a$ becomes (Fig. 86):

$$\sqrt{(x+c)^2+y^2} - \sqrt{(x-c)^2+y^2} = \pm 2a.$$

Squaring both members we find

$$x^2+y^2+c^2-\sqrt{(x^2+y^2+c^2-2\,cx)\,(x^2+y^2+c^2+2\,cx)}=2\,a^2;$$

squaring again and reducing as in § 206, we find exactly the same equation as in § 206:

$$(a^2-c^2)x^2+a^2y^2=a^2(a^2-c^2).$$

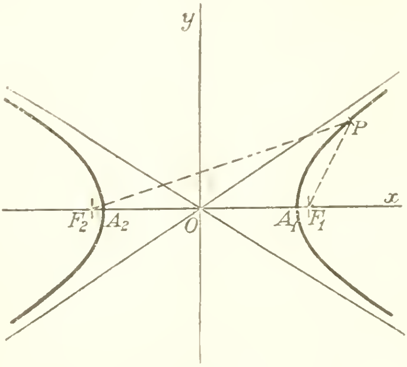


FIG. 86

But in the present case $c>a$, while for the ellipse we had $c<a$. We put, therefore, for the hyperbola

$$c^2-a^2=b^2;$$

the equation then reduces to the form

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1, \tag{3}$$

which is the ***cartesian equation*** *of the hyperbola referred to its axes.*

209. Properties of the Hyperbola. The equation (3) shows at once: (a) that the curve is symmetric to Ox and to Oy; (b) that the intercepts on the axis Ox are $\pm a$, and that the curve does not intersect the axis Oy.

The line F_2F_1 joining the foci and the perpendicular bisector of F_2F_1 are called the ***axes*** of the hyperbola; the intersection O of these axes of symmetry is called the ***center***.

The hyperbola has only two ***vertices***, viz. the intersections A_1, A_2 with the axis containing the foci.

The shape of the hyperbola is quite different from that of the ellipse. Solving the equation for y we have

$$y = \pm \frac{b}{a}\sqrt{x^2 - a^2}, \tag{4}$$

which shows that the curve extends to infinity from A_1 to the right and from A_2 to the left, but has no real points between the lines $x = a$, $x = -a$.

The line F_2F_1 containing the foci is called the ***transverse axis***: the perpendicular bisector of F_2F_1 is called the ***conjugate axis***. The lengths a, b are called the *transverse and conjugate semi-axes*.

In the particular case when $a = b$, the equation (3) reduces to

$$x^2 - y^2 = a^2,$$

and such a hyperbola is called ***rectangular*** or ***equilateral***.

210. Asymptotes. In sketching the hyperbola (3) or (4) it is best to draw first of all the two straight lines

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0,$$

i.e.

$$y = \pm \frac{b}{a}x, \tag{5}$$

which are called the ***asymptotes*** of the hyperbola.

Comparing with equation (4) it appears that, for any value of x, the ordinates of the hyperbola (4) are always (in absolute value) less than those of the lines (5); but the difference becomes less as x increases, approaching zero as x increases indefinitely.

Thus, the hyperbola approaches its asymptotes more and more closely, the farther we recede from the center on either side, without ever reaching these lines at any finite distance from the center.

EXERCISES

1. Sketch the hyperbola $x^2/16 - y^2/4 = 1$, after drawing the asymptotes, by determining a few points from the equation solved for y; mark the foci.

2. Sketch the rectangular hyperbola $x^2 - y^2 = 9$. Why the name rectangular?

3. With respect to the same axes draw the hyperbolas:

(*a*) $20x^2 - y^2 = 12$. (*b*) $x^2 - 20y^2 = 12$. (*c*) $x^2 - y^2 = 12$.

4. The equation $-x^2/a^2 + y^2/b^2 = 1$ represents a hyperbola whose foci lie on the axis Oy. Sketch the curves:

(*a*) $-3x^2 + 4y^2 = 24$. (*b*) $x^2 - 3y^2 + 18 = 0$. (*c*) $x^2 - y^2 + 16 = 0$.

5. Sketch to the same axes the hyperbolas:

$$\frac{x^2}{9} - y^2 = 1, \quad \frac{x^2}{9} - y^2 = -1.$$

Two such hyperbolas having the same asymptotes are called *conjugate*.

6. What happens to the hyperbola $x^2/a^2 - y^2/b^2 = 1$ as a varies? as b varies?

7. The equation $x^2/a^2 - y^2/b^2 = k$ represents a family of similar hyperbolas in which k is the parameter. What happens as k changes from 1 to -1? What members of this family are conjugate?

8. Find the foci of the hyperbolas:

(*a*) $9x^2 - 16y^2 = 144$. (*b*) $3x^2 - y^2 = 12$.

9. Find the hyperbola with foci $(0, 3)$, $(0, -3)$ and transverse axis 4.

10. Find the equation of the hyperbola referred to its axes when the distance between the vertices is one half the distance between the foci.

11. Find the distance from an asymptote to a focus of a hyperbola.

12. Show that the product of the distances from any point of a hyperbola to its asymptotes is constant.

13. Find the hyperbola through the point $(1, 1)$ with asymptotes

$$y = \pm 2x.$$

14. Find the equation of the hyperbola whose foci are $(1, 1)$, $(-1, -1)$, and transverse axis 2, and sketch the curve.

211. Ellipse as Projection of Circle. If a circle be turned about a diameter $A_2A_1 = 2a$ through an angle ϵ ($< \frac{1}{2}\pi$) and then projected on the original plane, the projection is an ellipse.

For, if in the original plane we take the center O as origin and OA_1 as axis Ox (Fig. 87), the ordinate QP of every point P of the projection is the projection of the corresponding ordinate QP_1 of the circle; *i.e.*

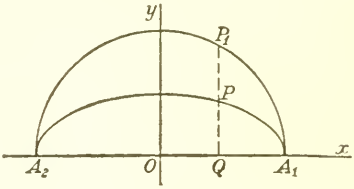


FIG. 87

$$QP = QP_1 \cos \epsilon.$$

The equation of the projection is therefore obtained from the equation

$$x^2 + y^2 = a^2$$

of the circle by replacing y by $y/\cos \epsilon$. The resulting equation

$$x^2 + \frac{y^2}{\cos^2 \epsilon} = a^2$$

represents an ellipse whose semi-axes are a, the radius of the circle, and $b = a \cos \epsilon$, the projection of this radius.

212. Construction of Ellipse from Circle. We have just seen that, if $a > b$, the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

can be obtained from its *circumscribed circle* $x^2 + y^2 = a^2$ by reducing all the ordinates of this circle in the ratio b/a. This also appears by comparing the ordinates

$$y = \pm \frac{b}{a}\sqrt{a^2 - x^2}$$

of the ellipse with the ordinates $y = \pm\sqrt{a^2 - x^2}$ of the circle.

But the same ellipse can also be obtained from its *inscribed circle* $x^2 + y^2 = b^2$ by increasing each abscissa in the ratio a/b, as appears at once by solving for x.

It follows that when the semi-axes a, b are given, points of the ellipse can be constructed by drawing concentric circles of radii a, b and a pair of perpendicular diameters (Fig. 88); if

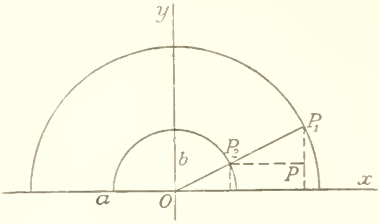


FIG. 88

any radius meets the circles at P_1, P_2, the intersection P of the parallels through P_1, P_2 to the diameters is a point of the ellipse.

213. Tangent to Ellipse. It follows from § 211 that if $P(x, y)$ is any point of the ellipse and P_1 that point of the circumscribed circle which has the same abscissa, *the tangents at P to the ellipse and at P_1 to the circle must meet at a point T on the major axis* (Fig. 89).

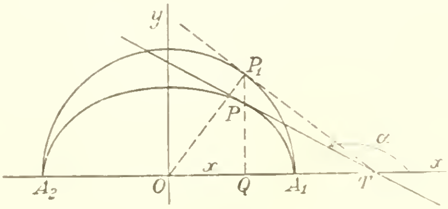


FIG. 89

For, as the circle is turned about A_2A_1 into the position in which P is the projection of P_1, the tangent to the circle at P_1 is turned into the position whose projection is PT, the point T on the axis remaining fixed.

The tangent $x_1X + y_1Y = a^2$ to the circle at $P_1(x_1, y_1)$ meets the axis Ox at the point T whose abscissa is

$$OT = a^2/x_1 = a^2/x.$$

Hence the equation of the tangent at $P(x, y)$ to the ellipse is

$$\begin{vmatrix} X & Y & 1 \\ x & y & 1 \\ \dfrac{a^2}{x} & 0 & 1 \end{vmatrix} = 0,$$

i.e.
$$yX - \left(x - \frac{a^2}{x}\right)Y - a^2\frac{y}{x} = 0;$$

dividing by a^2y/x and observing that, by the equation of the ellipse, $x^2 - a^2 = -(a^2/b^2)y^2$ we find

(6)
$$\frac{xX}{a^2} + \frac{yY}{b^2} = 1$$

as *equation of the tangent to the ellipse*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

at the point $P(x, y)$.

214. Slope of Ellipse. It follows from the equation of the tangent that the ***slope*** of the ellipse at any point $P(x, y)$ is

$$\tan\alpha = -\frac{b^2}{a^2}\frac{x}{y}.$$

The slope being the derivative y' can be found more directly by differentiating the equation (1) of the ellipse (remembering that y is a function of x, compare §§ 181–185); this gives

$$\frac{2x}{a^2} + \frac{2yy'}{b^2} = 0,$$

whence

$$y' = \tan\alpha = -\frac{b^2}{a^2}\frac{x}{y}.$$

The equation (6) of the tangent is readily derived from this value of the slope.

215. Eccentricity. For the length of the *focal radius* F_1P of any point $P(x, y)$ of the ellipse (1) we have (Fig. 90), since $a^2 - b^2 = c^2$:

$$F_1P^2=(x-c)^2+y^2=(x-c)^2+\frac{b^2}{a^2}(a^2-x^2)=\frac{1}{a^2}(a^4-2a^2cx+c^2x^2),$$

whence
$$F_1P=\pm\left(a-\frac{c}{a}x\right).$$

The ratio c/a of the distance $2c$ of the foci to the major axis $2a$ is called the (numerical) ***eccentricity*** of the ellipse. Denoting it by e we have

$$F_1P=\pm(a-ex),$$

and similarly we find

$$F_2P=\pm(a+ex).$$

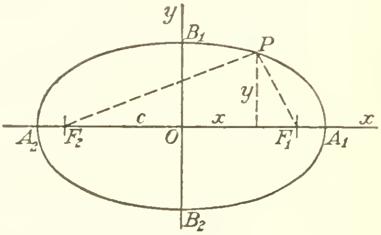


FIG. 90

For the hyperbola (3) we find in the same way, if we again put $e = c/a$, exactly the same expressions for the focal radii F_1P, F_2P (in absolute value). But as for the ellipse $c^2 = a^2 - b^2$ while for the hyperbola $c^2 = a^2 + b^2$ it follows that *the eccentricity of the ellipse is always a proper fraction becoming zero only for a circle, while the eccentricity of the hyperbola is always greater than one.*

216. Equation of Normal to Ellipse. As the normal to a curve is the perpendicular to its tangent through the point of contact, the *equation of the normal* to the ellipse (1) at the point $P(x, y)$ is readily found from the equation (6) of the tangent as

$$\frac{y}{b^2}X-\frac{x}{a^2}Y=xy\left(\frac{1}{b^2}-\frac{1}{a^2}\right)=\frac{c^2}{a^2b^2}xy,$$

i.e.
$$\frac{a^2}{x}X-\frac{b^2}{y}Y=c^2.$$

The intercept made by this normal on the axis Ox is therefore

$$ON = \frac{c^2}{a^2}x = e^2x.$$

From this result it appears by § 215 that (Fig. 91)

$$F_2N = c + e^2x = e(a + ex) = e \cdot F_2P,$$
$$F_1N = c - e^2x = e(a - ex) = e \cdot F_1P;$$

hence the normal divides the distance F_2F_1 in the ratio of the adjacent sides F_2P, F_1P of the triangle F_1PF_2. It follows that *the normal bisects the angle between the focal radii* PF_1, PF_2; in other words, the focal radii are equally inclined to the tangent.

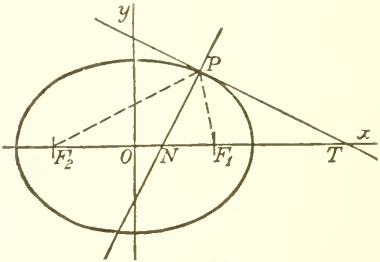


FIG. 91

217. Construction of any Hyperbola from Rectangular Hyperbola. The ordinates (4),

$$y = \pm \frac{b}{a}\sqrt{x^2 - a^2},$$

of the hyperbola (3) are b/a times the corresponding ordinates

$$y = \pm \sqrt{x^2 - a^2}$$

of the equilateral hyperbola (end of § 209) having the same transverse axis. When $b < a$, we can put $b/a = \cos \epsilon$ and regard the general hyperbola as the projection of the equilateral hyperbola of equal transverse axis. When $b > a$, we can put $a/b = \cos \epsilon$ so that the equilateral hyperbola can be regarded as the projection of the general hyperbola.

In either case it is clear that the tangents to the general and equilateral hyperbolas at corresponding points (*i.e.* at points having the same abscissa) must intersect on the axis Ox.

218. Slope of Equilateral Hyperbola. To find the slope of the equilateral hyperbola

$$x^2 - y^2 = a^2,$$

observe that the slope of any secant joining the point $P(x, y)$ and $P_1(x_1, y_1)$ is $(y_1 - y)/(x_1 - x)$, and that the relations

$$y^2 = x^2 - a^2,$$
$$y_1^2 = x_1^2 - a^2$$

give $\quad y^2 - y_1^2 = x^2 - x_1^2$, *i.e.* $(y - y_1)(y + y_1) = (x - x_1)(x + x_1)$,

whence $$\frac{y - y_1}{x - x_1} = \frac{x + x_1}{y + y_1}.$$

Hence, in the limit when P_1 comes to coincidence with P, we find for the *slope of the tangent* at $P(x, y)$:

$$\tan \alpha = \frac{x}{y}.$$

The equation of the tangent to the equilateral hyperbola is therefore

$$Y - y = \frac{x}{y}(X - x),$$

i.e. since $x^2 - y^2 = a^2$:

$$xX - yY = a^2.$$

219. Tangent to the Hyperbola. It follows as in § 213 that the *tangent to the general hyperbola* (3) has the equation

$$\frac{xX}{a^2} - \frac{yY}{b^2} = 1. \tag{7}$$

The slope of the hyperbola (3) is therefore

$$\tan \alpha = \frac{b^2}{a^2}\frac{x}{y}.$$

This slope might of course have been obtained directly by differentiating the equation (3) (compare § 214).

Notice that the equations (6), (7) of the tangents are obtained from the equations (1), (3) of the curves by replacing x^2, y^2 by xX, yY, respectively (compare §§ 89, 186).

It is readily shown (compare § 216) that for the hyperbola (3) the tangent meets the axis Ox at the point T that divides the distance of the foci F_2F_1 proportionally to the focal radii F_2P, F_1P, so that *the tangent to the hyperbola bisects the angle between the focal radii.*

EXERCISES

1. Show that a right cylinder whose cross-section (*i.e.* section at right angles to the generators) is an ellipse of semi-axes a, b has two (oblique) circular sections of radius a; find their inclinations to the cross-section.

2. Derive the equation of the normal to the hyperbola (3).

3. Find the polar equations of the ellipse and hyperbola, with the center as pole and the major (transverse) axis as polar axis.

4. Find the lengths of the tangent, subtangent, normal, and subnormal in terms of the coordinates at any point of the ellipse.

5. Show that an ellipse and hyperbola with common foci are orthogonal.

6. Show that the eccentricity of a hyperbola is equal to the secant of half the angle between the asymptotes.

7. Express the cosine of the angle between the asymptotes of a hyperbola in terms of its eccentricity.

8. Show that the tangents at the vertices of a hyperbola intersect the asymptotes at points on the circle about the center through the foci.

9. Show that the point of contact of a tangent to a hyperbola is the midpoint between its intersections with the asymptotes.

10. Show that the area of the triangle formed by the asymptotes and any tangent to a hyperbola is constant.

11. Show that the product of the distances from the center of a hyperbola to the intersections of any tangent with the asymptotes is constant.

12. Show that the tangent to a hyperbola at any point bisects the angle between the focal radii of the point.

13. As the sum of the focal radii of every point of an ellipse is constant (§ 204) and the normal bisects the angle between the focal radii (§ 216), a sound wave issuing from one focus is reflected by the ellipse to the other focus. This is the explanation of "whispering galleries." Find the semi-axes of an elliptic gallery in which sound is reflected from one focus to the other at a distance of 69 ft. in 1/10 sec. (the velocity of sound is 1090 ft./sec.).

14. Show that the distance from any point of an equilateral hyperbola to its center is a mean proportional to the focal radii of the point.

15. Show that the bisector of the angle formed by joining any point of an equilateral hyperbola to its vertices is parallel to an asymptote.

16. For the ellipse obtained by turning a circle of radius a about a diameter through an angle ϵ and projecting it on the plane of the circle, show that the distance between the foci is $= 2a \sin \epsilon$; in particular, show that the foci of a circle are at the center.

17. Show that the tangents at the extremities of any *diameter* (chord through the center) of an ellipse or hyperbola are parallel.

18. Let the normal at any point P of an ellipse referred to its axes cut the coordinate axes at Q and R; find the ratio PQ/PR.

19. Show that a tangent at any point of the circle circumscribed about an ellipse is also a tangent to the circle with center at a focus and radius equal to the focal radius of the corresponding point of the ellipse.

20. Show that the lines joining any point of an ellipse to the ends of the minor axis intersect the major axis (produced) in points inverse with respect to the circumscribed circle.

21. Show that the product of the y-intercept of the tangent at any point of an ellipse and the ordinate of the point of contact is constant.

22. Show that the normals to an ellipse through its intersections with a circle determined by a given point of the minor axis and the foci pass through the given point.

23. Find the locus of the center of a circle which touches two fixed non-intersecting circles.

24. Find the locus of a point at which two sounds emitted at an interval of one second at two points 2000 ft. apart are heard simultaneously.

220. Intersections of a Straight Line and an Ellipse. The intersections of the ellipse (1) with any straight line are found by solving the simultaneous equations

$$b^2x^2 + a^2y^2 = a^2b^2,$$
$$y = mx + k.$$

Eliminating y, we find a quadratic equation in x:

$$(m^2a^2 + b^2)x^2 + 2\,mka^2x + (k^2 - b^2)a^2 = 0.$$

To each of the two roots the corresponding value of y results from the equation $y = mx + k$.

Thus, *a straight line can intersect an ellipse in not more than two points.*

221. Slope Form of Tangent Equations. If the roots of the quadratic equation are equal, the line has but one point in common with the ellipse and is a tangent.

The condition for equal roots is

$$m^2k^2a^2 = (m^2a^2 + b^2)(k^2 - b^2),$$

whence $$k = \pm\sqrt{m^2a^2 + b^2}.$$

The two parallel lines

(8) $$y = mx \pm \sqrt{m^2a^2 + b^2}$$

are therefore tangents to the ellipse (1), whatever the value of m. This equation is called the *slope form* of the equation of a tangent to the ellipse.

It can be shown in the same way that a straight line cannot intersect a hyperbola in more than two points, and that the two parallel lines

$$y = mx \pm \sqrt{m^2a^2 - b^2}$$

have each but one point in common with the hyperbola (3).

222. The condition that a line be a tangent to an ellipse or hyperbola assumes a simple form also when the line is given in the general form

$$Ax + By + C = 0.$$

Substituting the value of y obtained from this equation in the equation (1) of the ellipse, we find for the abscissas of the points of intersection the quadratic equation:

$$(A^2a^2 + B^2b^2)x^2 + 2\,ACa^2x + (C^2 - B^2b^2)a^2 = 0;$$

the condition for equal roots is

$$A^2C^2a^2 = (A^2a^2 + B^2b^2)(C^2 - B^2b^2),$$

which reduces to

$$A^2a^2 + B^2b^2 = C^2.$$

The line is therefore a tangent whenever this condition is satisfied.

When the line is given in the normal form,

$$x\cos\beta + y\sin\beta = p.$$

the condition becomes

$$p^2 = a^2\cos^2\beta + b^2\sin^2\beta.$$

223. Tangents from an Exterior Point. By § 221 the line

$$y = mx + \sqrt{m^2a^2 + b^2}$$

is tangent to the ellipse (1) whatever the value of m. The condition that this line pass through any given point (x_1, y_1) is

$$y_1 = mx_1 + \sqrt{m^2a^2 + b^2};$$

transposing the term mx_1, and squaring, we find the following quadratic equation for m:

$$m^2x_1^2 - 2\,mx_1y_1 + y_1^2 = m^2a^2 + b^2,$$

i.e.

$$(x_1^2 - a^2)m^2 - 2\,x_1y_1m + y_1^2 - b^2 = 0.$$

The roots of this equation are the slopes of those lines through (x_1, y_1) that are tangent to the ellipse (1).

Thus, *not more than two tangents can be drawn to an ellipse from any point.* Moreover, these tangents are real and different, real and coincident, or imaginary, according as

$$x_1^2y_1^2 \gtreqless (x_1^2 - a^2)(y_1^2 - b^2).$$

This condition can also be written in the form

$$b^2x_1^2 + a^2y_1^2 \gtreqless a^2b^2,$$

i.e.

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1 \gtreqless 0.$$

Hence, to see whether real tangents can be drawn from a point (x_1, y_1) to the ellipse (1) we have only to substitute the coordinates of the point for x, y in the expression

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1;$$

if the expression is zero, the point (x_1, y_1) lies on the ellipse, and only one tangent is possible; if the expression is positive, two real tangents can be drawn, and the point is said to lie ***outside*** the ellipse; if the expression is negative, no real tangents exist, and the point is said to lie ***within*** the ellipse.

These definitions of inside and outside agree with what we would naturally call the inside or outside of the ellipse. But the whole discussion applies equally to the hyperbola (3) where the distinction between inside and outside is not so obvious.

224. Symmetry. Since the ellipse, as well as the hyperbola, has two rectangular axes of symmetry, the *axes* of the curve, it has a ***center***, the intersection of these axes, *i.e.* a point of symmetry such that every chord through this point is bisected at this point (compare § 135). Analytically this means that since the equation (1), as well as (3), is not changed by replacing x by $-x$, nor by replacing y by $-y$, it is not changed by replacing both x and y by $-x$ and $-y$, respectively. In other words, if (x, y) is a point of the curve, so is $(-x, -y)$. This fact is expressed by saying that the origin is a point of symmetry, or center.

225. Conjugate Diameters. Any chord through the center of an ellipse or hyperbola is called a ***diameter*** of the curve.

Just as in the case of the circle, so for the ellipse *the locus of the midpoints of any system of parallel chords is a diameter.* This follows from the corresponding property of the circle because the ellipse can be regarded as the projection of a circle (§ 211). But this diameter is in general not perpendicular to the parallel chords; it is said to be ***conjugate*** to the diameter that occurs among the parallel chords. Thus, in Fig. 92, $P'Q'$ is conjugate to PQ (and *vice versa*).

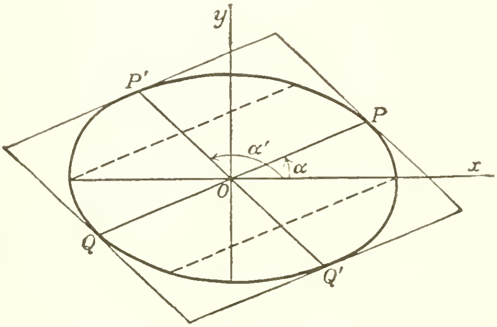


FIG. 92

To find the diameter conjugate to a given diameter $y = mx$ of the ellipse (1), let $y = mx + k$ be any parallel to the given diameter. If this parallel intersects the ellipse (1) at the real points (x_1, y_1) and (x_2, y_2), the midpoint has the coordinates $\frac{1}{2}(x_1 + x_2)$, $\frac{1}{2}(y_1 + y_2)$. The quadratic equation of § 220 gives

$$x = \frac{1}{2}(x_1 + x_2) = -\frac{ma^2k}{m^2a^2 + b^2}.$$

If instead of eliminating y we eliminate x, we obtain the quadratic equation

$$(m^2a^2 + b^2)y^2 - 2\,kb^2y + (k^2 - m^2a^2)b^2 = 0,$$

whence

$$y = \frac{1}{2}(y_1 + y_2) = \frac{b^2k}{m^2a^2 + b^2}.$$

Eliminating k between these results, we find the equation of the locus of the midpoints of the parallel chords of slope m:

(9) $$y = -\frac{b^2}{ma^2}x.$$

If $m = \tan\alpha$ is the slope of any diameter of the ellipse (1), *the slope of the conjugate diameter is*

$$m_1 = \tan\alpha_1 = -\frac{b^2}{ma^2}.$$

The diameter conjugate to this diameter of slope m_1 has therefore the slope

$$m_2 = -\frac{b^2}{m_1 a^2} = -\frac{b^2}{\left(-\frac{b^2}{ma^2}\right)a^2} = m;$$

i.e. it is the original diameter of slope m (Fig. 92). In other words, either one of the diameters of slopes m and m_1 is conjugate to the other; each bisects the chords parallel to the other.

226. Tangents Parallel to Diameters. Among the parallel lines of slope m, $y = mx + k$, there are two tangents to the ellipse, viz. (§ 221) those for which

$$k = \pm\sqrt{m^2a^2 + b^2},$$

their points of contact lie on (and hence determine) the conjugate diameter. This is obvious geometrically; it is readily verified analytically by showing that the coordinates of the intersections of the diameter of slope $-b^2/ma^2$ with the ellipse (1) satisfy the equations of the tangents of slope m, viz.

$$y = mx \pm\sqrt{m^2a^2 + b^2}.$$

The tangents at the ends of the diameter of slope m must of course be parallel to the diameter of slope m_1. The four tangents at the extremities of any two conjugate diameters thus form a circumscribed parallelogram (Fig. 92).

The diameter conjugate to either axis of the ellipse is the other axis; the parallelogram in this case becomes a rectangle.

227. Diameters of a Hyperbola. For the hyperbola the same formulas can be derived except that b^2 is replaced throughout by $-b^2$. But the geometrical interpretation is somewhat different because a line $y = mx$ meets the hyperbola (3) in real points only when $m < b/a$.

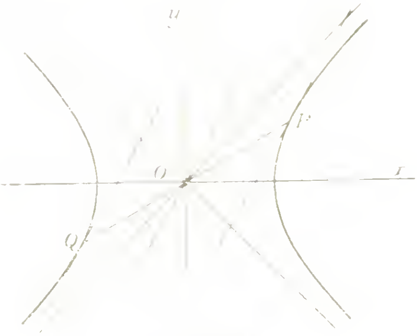


Fig. 93

The solution of the simultaneous equations

$$y = mx, \quad b^2x^2 - a^2y^2 = a^2b^2$$

gives:

$$x = \pm\frac{ab}{\sqrt{b^2 - m^2a^2}}, \quad y = \pm\frac{mab}{\sqrt{b^2 - m^2a^2}}.$$

These values are real if $m < b/a$ and imaginary if $m > b/a$ (Fig. 93). In the former case it is evidently proper to call the distance PQ between the real points of intersection a *diameter* of the hyperbola; its length is

$$PQ = 2\sqrt{x^2 + y^2} = 2ab\sqrt{\frac{1 + m^2}{b^2 - m^2a^2}}.$$

If $m > b/a$, this quantity is imaginary; but it is customary to speak even in this case of a diameter, its length being defined as the real quantity

$$2ab\sqrt{\frac{1 + m^2}{m^2a^2 - b^2}}.$$

By this convention the analogy between the properties of the ellipse and hyperbola is preserved.

228. Conjugate Diameters of a Hyperbola. Two diameters of the hyperbola are called ***conjugate*** if their slopes m, m_1 are such that

$$mm_1 = \frac{b^2}{a^2}.$$

One of these lines evidently meets the curve in real points, the other does not.

If $m < b/a$, the line $y = mx$, as well as any parallel line, meets the hyperbola (3) in two real points, and the locus of the midpoints of the chords parallel to $y = mx$ is found to be the diameter conjugate to $y = mx$, viz.

$$y = m_1x = \frac{b^2}{ma^2}x.$$

If $m > b/a$, the coordinates x_1, y_1 and x_2, y_2 of the intersections of $y = mx$ with the hyperbola are imaginary; but the arithmetic means $\frac{1}{2}(x_1 + x_2)$, $\frac{1}{2}(y_1 + y_2)$ are real, and the locus of the points having these coordinates is the real line

$$y = m_1x = \frac{b^2}{ma^2}x.$$

It may finally be noted that what was in § 227 defined as the length of a diameter that does not meet the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

in real points is the length of the real diameter of the hyperbola

$$-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1;$$

two such hyperbolas are called *conjugate*.

229. Parameter Equations. Eccentric Angle. Just as the parameter equations of the circle $x^2 + y^2 = a^2$ are (§ 194):

$$x = a \cos\theta, \quad y = a \sin\theta,$$

so those of the ellipse (1) are

$$x = a \cos\theta, \quad y = b \sin\theta,$$

and those of the hyperbola (3) are

$$x = a \sec\theta, \quad y = b \tan\theta.$$

In each case the elimination of the parameter θ (by squaring and then adding or subtracting) leads to the cartesian equation.

The angle θ, in the case of the circle, is simply the polar angle of the point $P(x, y)$. In the case of the ellipse, as appears from Fig. 94 (compare § 212), θ is the polar angle not of the point $P(x, y)$ of the ellipse, but of that point P_1 of the circumscribed circle which has the same abscissa as P, and also of that point P_2 of the inscribed circle which has the same ordinate as P. This angle $\theta = xOP_1$ is called the ***eccentric angle*** of the point $P(x, y)$ of the ellipse.

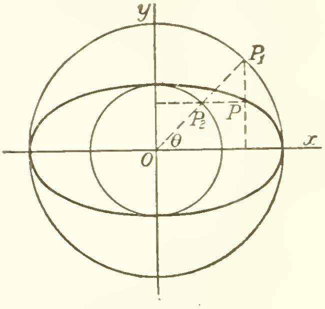


Fig. 94

In the case of the hyperbola the eccentric angle θ determines the point $P(x, y)$ as follows (Fig. 95). Let a line through O inclined at the angle θ to the transverse axis meet the circle of radius a about the center at A, and let the transverse axis meet the circle of radius b about the center at B. Let the tangent at A meet the transverse axis at A' and the tangent at B meet the line OA at B'. Then the parallels to the axes through A' and B' meet at P.

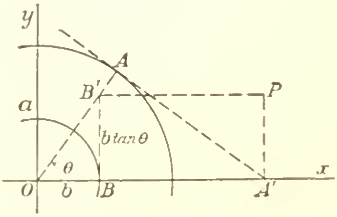


Fig. 95

230. Area of Ellipse. Since any ellipse of semi-axes a, b can be regarded as the projection of a circle of radius a, inclined to the plane of the ellipse at an angle ϵ such that $\cos \epsilon = b/a$, the area A of the ellipse is $A = \pi a^2 \cos \epsilon = \pi ab$.

EXERCISES

1. Find the tangents to the ellipse $x^2 + 4y^2 = 16$, which pass through the following points:

(a) $(2, \sqrt{3})$, (b) $(-3, \frac{1}{2}\sqrt{7})$, (c) $(4, 0)$, (d) $(-8, 0)$.

2. Find the tangents to the hyperbola $2x^2 - 3y^2 = 18$, which pass through the following points:

(a) $(-6, 3\sqrt{2})$, (b) $(-3, 0)$, (c) $(4, -\sqrt{5})$, (d) $(0, 0)$.

3. Find the intersections of the line $x - 2y = 7$ and the hyperbola

$$x^2 - y^2 = 5.$$

4. Find the intersections of the line $3x + y - 1 = 0$ and the ellipse

$$x^2 + 4y^2 = 65.$$

5. For what value of k will the line $y = 2x + k$ be a tangent to the hyperbola $x^2 - 4y^2 - 4 = 0$?

6. For what values of m will the line $y = mx + 2$ be tangent to the ellipse $x^2 + 4y^2 - 1 = 0$?

7. Find the conditions that the following lines are tangent to the hyperbola $x^2/a^2 - y^2/b^2 = 1$:

(a) $Ax + By + C = 0$, (b) $x \cos \beta + y \sin \beta = p$.

8. Are the following points on, outside, or inside the ellipse $x^2 + 4y^2 = 4$?

(a) $(\frac{3}{2}, \frac{3}{4})$, (b) $(\frac{7}{4}, -\frac{1}{8})$, (c) $(-\frac{1}{4}, -\frac{3}{4})$.

9. Are the following points on, outside, or inside the hyperbola $9x^2 - y^2 = 9$? (a) $(\frac{5}{4}, -\frac{9}{4})$, (b) $(1.35, 2.15)$, (c) $(1.3, 2.6)$.

10. Find the difference of the eccentric angles of points at the extremities of conjugate diameters of an ellipse.

11. Show that conjugate diameters of an equilateral hyperbola are equal.

12. Show that an asymptote is its own conjugate diameter.

13. Show that the segments of any line between a hyperbola and its asymptotes are equal.

14. Find the tangents to an ellipse referred to its axes which have equal intercepts.

15. What is the greatest possible number of normals that can be drawn from a given point to an ellipse or hyperbola?

16. Show that tangents drawn at the extremities of any chord of an ellipse (or hyperbola) intersect on the diameter conjugate to the chord.

17. Show that lines joining the extremities of the axes of an ellipse are parallel to conjugate diameters.

18. Show that chords drawn from any point of an ellipse to the extremities of a diameter are parallel to conjugate diameters.

19. Find the product of the perpendiculars let fall to any tangent from the foci of an ellipse (or hyperbola).

20. The earth's orbit is an ellipse of eccentricity .01677 with the sun at a focus. The mean distance (major semi-axis) between the sun and earth is 93 million miles. Find the distance from the sun to the center of the orbit.

21. Find the sum of the squares of any two conjugate semi-diameters of an ellipse. Find the difference of the squares of conjugate semi-diameters of a hyperbola.

22. Find the area of the parallelogram circumscribed about an ellipse with sides parallel to any two conjugate diameters.

23. Find the angle between conjugate diameters of an ellipse in terms of the semi-diameters and semi-axes.

24. Express the area of a triangle inscribed in an ellipse referred to its axes in terms of the eccentric angles of the vertices.

25. The circle which is the locus of the intersection of two perpendicular tangents to an ellipse or hyperbola is called the *director-circle* of the conic. Find its equation: (a) For the ellipse. (b) For the hyperbola.

26. Find the locus of a point such that the product of its distances from the asymptotes of a hyperbola is constant. For what value of this constant is the locus the hyperbola itself?

27. Find the locus of the intersection of normals drawn at corresponding points of an ellipse and the circumscribed circle.

28. Two points A, B of a line l whose distance is $AB = a$ move along two fixed perpendicular lines; find the path of any point P of l.

CHAPTER XI

CONIC SECTIONS — EQUATION OF SECOND DEGREE

PART I. DEFINITION AND CLASSIFICATION

231. Conic Sections. The ellipse, hyperbola, and parabola are together called ***conic sections***, or simply ***conics***, because the curve in which a right circular cone is intersected by any plane (not passing through the vertex) is an ellipse or hyperbola according as the plane cuts only one of the half-cones or both, and is a parabola when the plane is parallel to a generator of the cone. This will be proved and more fully discussed in §§ 239–243.

232. General Definition. The three conics can also be defined by a common property in the plane: *the locus of a point for which the ratio of its distances from a fixed point and from a fixed line is constant is a conic*, viz. an ***ellipse*** if the constant ratio is less than one, a ***hyperbola*** if the ratio is greater than one, and a ***parabola*** if the ratio is equal to one.

We shall find that this constant ratio is equal to the ***eccentricity*** $e = c/a$ as defined in § 215. Just as in the case of the parabola for which the above definition agrees with that of § 172, we shall call the fixed line d_1 ***directrix***, and the fixed point F_1 ***focus*** (Fig. 96).

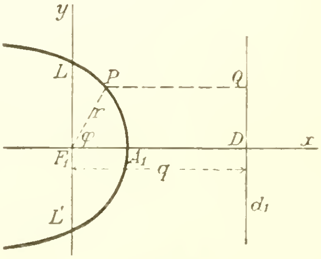


Fig. 96

233. Polar Equation. Taking the focus F_1 as pole, the perpendicular from F_1 toward the directrix d_1 as polar axis, and putting the given distance $F_1D = q$, we have $F_1P = r$, $PQ = q - r\cos\phi$, r and ϕ being the polar coordinates of any point P of the conic. The condition to be satisfied by the point P, viz. $F_1P/PQ = e$, *i.e.* $F_1P = e \cdot PQ$ becomes, therefore,

$$r = e(q - r\cos\phi),$$

whence $$r = \frac{eq}{1 + e\cos\phi}.$$

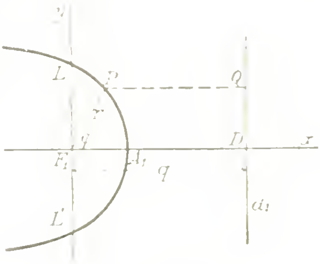


FIG. 96

This then is the ***polar equation of a conic*** *if the focus is taken as pole and the perpendicular from the focus toward the directrix as polar axis.*

It is assumed that the distance q between the fixed point and fixed line is not zero; the ratio e, *i.e.* the eccentricity of the conic, may be any positive number.

234. Plotting the Conic. By means of this polar equation the conic can be plotted by points when e and q are given. Thus, for $\phi = 0$ and $\phi = \pi$, we find $eq/(1 + e)$ and $eq/(1 - e)$ as the intercepts F_1A_1 and F_1A_2 on the polar axis; A_1, A_2 are the vertices. For any negative value of ϕ (between 0 and $-\pi$) the radius vector has the same length as for the same positive value of ϕ. The segment LL' cut off by the conic on the perpendicular to the polar axis drawn through the pole is called the ***latus rectum***; its length is $2eq$. Notice that in the ellipse and hyperbola, *i.e.* when $e \neq 1$, the vertex A_1 does not bisect the distance F_1D (as it does in the parabola), but that

$$F_1A_1/A_1D = e.$$

If in Fig. 96, other things being equal, the sense of the polar axis be reversed, we obtain Fig. 97. We have again $F_1P = r$; but the distance of P from the directrix d_1 is $QP = q + r \cos \phi$, so that the polar equation of the conic is now:

$$r = \frac{eq}{1 - e \cos \phi}.$$

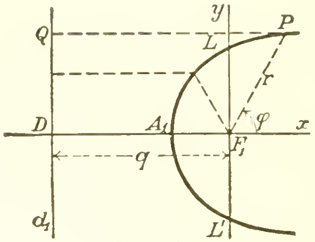


FIG. 97

235. Classification of Conics. For $e = 1$, the equations of §§ 233–234 reduce to the equations of the parabola given in §§ 172, 173. It remains to show that for $e < 1$ and $e > 1$ these equations represent respectively an ellipse and a hyperbola as defined in §§ 204, 207.

To show this we need only introduce cartesian coordinates and then transform to the *center*, *i.e.* to the midpoint O between the intersections A_1, A_2 of the curve with the polar axis.

236. Transformation to Cartesian Coordinates. The equation of § 233,

$$r = e(q - r \cos \phi)$$

becomes in cartesian coordinates, with the pole F_1 as origin and the polar axis as axis Ox (Fig. 96):

$$\sqrt{x^2 + y^2} = e(q - x),$$

or rationalized:

$$(1 - e^2)x^2 + 2\, e^2qx + y^2 = e^2q^2.$$

The midpoint O between the vertices A_1, A_2 at which the curve meets the axis Ox has, by § 234, the abscissa

$$\tfrac{1}{2}\, eq\left(\frac{1}{1+e} - \frac{1}{1-e}\right) = -\frac{e^2q}{1-e^2};$$

this also follows from the cartesian equation, with $y = 0$.

237. Change of Origin to Center. To transform to parallel axes through this point O we have to replace x by $x - e^2q/(1-e^2)$; the equation in the new coordinates is therefore

$$(1-e^2)\left(x - \frac{e^2q}{1-e^2}\right)^2 + 2\,e^2q\left(x - \frac{e^2q}{1-e^2}\right) + y^2 = e^2q^2,$$

and this reduces to

$$(1-e^2)x^2 + y^2 = e^2q^2\left(1 + \frac{e^2}{1-e^2}\right) = \frac{e^2q^2}{1-e^2},$$

i.e.

$$\frac{x^2}{\frac{e^2q^2}{(1-e^2)^2}} + \frac{y^2}{\frac{e^2q^2}{1-e^2}} = 1.$$

If $e < 1$ this is an ellipse with semi-axes

$$a = \frac{eq}{1-e^2}, \quad b = \frac{eq}{\sqrt{1-e^2}};$$

if $e > 1$ it is a hyperbola with semi-axes

$$a = \frac{eq}{e^2-1}, \quad b = \frac{eq}{\sqrt{e^2-1}}.$$

238. Focus and Directrix. The distance c (in absolute value) from the center O to the focus F_1 is, as shown above, for the ellipse

$$c = \frac{e^2q}{1-e^2} = ae,$$

for the hyperbola

$$c = \frac{e^2q}{e^2-1} = ae.$$

The distance (in absolute value) of the directrix from the center O is for the ellipse, since $q = a(1-e^2)/e = a/e - ae$:

$$OD = c + q = ae + \frac{a}{e} - ae = \frac{a}{e}.$$

and for the hyperbola, since $q = ae - a/e$:

$$OD = c - q = ae - ae + \frac{a}{e} = \frac{a}{e}.$$

It is clear from the symmetry of the ellipse and hyperbola that each of these curves has two foci, one on each side of the center at the distance ae from the center, and two directrices whose equations are $x = \pm a/e$.

EXERCISES

1. Sketch the following conics:

$(a)\ r = \dfrac{6}{2 + 3\cos\phi}, \qquad (b)\ r = \dfrac{2}{2 + \cos\phi}, \qquad (c)\ r = \dfrac{1}{1 - 2\cos\phi}.$

2. Sketch the following conics and find their foci and directrices.

$(a)\ x^2 + 4y^2 = 4, \qquad (b)\ 4x^2 + y^2 = 4,$

$(c)\ x^2 - 4y^2 = 4, \qquad (d)\ 4x^2 - y^2 = 4,$

$(e)\ 16x^2 + 25y^2 = 400, \qquad (f)\ 9x^2 - 16y^2 = 144,$

$(g)\ 9x^2 - 16y^2 + 144 = 0, \qquad (h)\ x^2 - y^2 = 2.$

3. Show that the following equations represent ellipses or hyperbolas and find their centers, foci, and directrices:

$(a)\ x^2 + 3y^2 - 2x + 6y + 1 = 0, \qquad (b)\ 12x^2 - 4y^2 - 12x - 9 = 0,$

$(c)\ 5x^2 + y^2 + 20x + 15 = 0, \qquad (d)\ 5x^2 - 4y^2 + 8y + 16 = 0.$

4. Find the length of the latus rectum of an ellipse and a hyperbola in terms of the semi-axes.

5. Show that the intersections of the tangents at the vertices with the asymptotes of a hyperbola lie on the circle about the center passing through the foci.

6. Show that when tangents to an ellipse or hyperbola are drawn from any point of a directrix the line joining the points of contact passes through a focus.

7. From the definition (§ 232) of an ellipse and hyperbola, show that the sum and difference respectively of the focal radii of any point of the conic is constant.

8. Find the locus of the midpoints of chords drawn from one end of: (*a*) the major axis of an ellipse; (*b*) the minor axis.

9. The eccentricity of an ellipse with one focus and corresponding directrix fixed is allowed to vary; show that the locus of the ends of the minor axis is a parabola.

10. Find the locus of § 232 when the fixed point lies on the fixed line.

239. The Conics as Sections of a Cone. As indicated by their name the conic sections, *i.e.* the parabola, ellipse, and hyperbola, can be defined as the curves in which a right circular cone is cut by a plane (§ 231).

In Figs. 98, 99, 100, V is the vertex of the cone, $\measuredangle CVC' = 2\alpha$ the angle at its vertex; OQ indicates the cutting plane, CVC' that plane through the axis of the cone which is perpendicular to the cutting plane. The intersection OQ of these two planes is evidently an axis of symmetry for the conic.

The conic is a parabola, ellipse, or hyperbola, according as OQ is parallel to the generator VC' of the cone (Fig. 98), meets VC' at a point O' belonging to the same half-cone as does O (Fig. 99), or meets VC' at a point O' of the other half-cone (Fig. 100). If the angle COQ be called β, the conic is

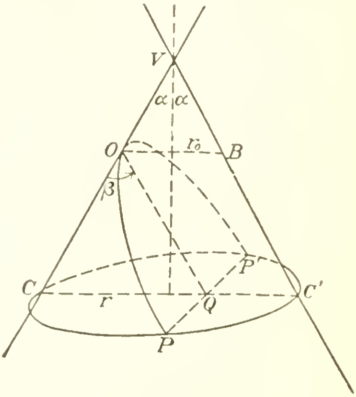


FIG. 98

a parabola if $\beta = 2\alpha$ (Fig. 98),
an ellipse if $\beta > 2\alpha$ (Fig. 99),
a hyperbola if $\beta < 2\alpha$ (Fig. 100).

In each of the three figures CC' represents the diameter $2r$ of any cross-section of the cone (*i.e.* of any section at right angles to its axis). We take O as origin, OQ as axis Ox, so that (Fig. 98) $OQ = x$, $QP = y$ are the coordinates of any point P of the conic.

As QP is the ordinate of the circular cross-section $CPC'P'$ we have in each of the three cases:

$$y^2 = QP^2 = CQ \cdot QC'.$$

240. Parabola. In the first case (Fig. 98), when $\beta = 2\alpha$ so that OQ is parallel to VC', the expression

$$\frac{y^2}{x} = \frac{QP^2}{OQ} = \frac{CQ}{OQ} \cdot QC'$$

is constant, *i.e.* the same at whatever distance from the vertex we may take the cross-section $CPC'P'$. For, QC' is equal to the diameter $OB = 2r_0$ of the cross-section through O, and

$$CQ/OQ = CC'/VC = 2\,r/r\csc\alpha = 2\sin\alpha.$$

Hence, denoting the constant $r_0 \sin\alpha$ by p we have

$$\frac{CQ}{OQ} \cdot QC' = 4\,r_0 \sin\alpha = 4\,p.$$

The equation of the conic in this case, referred to its axis OQ and vertex O, is therefore

$$y^2 = 4\,px.$$

Notice that as $p = r_0 \sin\alpha$ the focus is found as the foot of the perpendicular from the midpoint of OB on OQ.

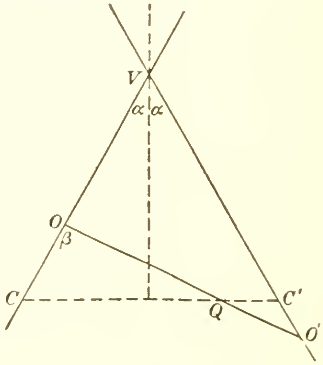


FIG. 99

241. Ellipse. In the second case (Fig. 99), *i.e.* when $\beta > 2\alpha$, if we put

$$OO' = 2\,a,$$

it can be shown that

$$\frac{y^2}{x(2\,a - x)} = \frac{QP^2}{OQ \cdot QO'}$$

is constant. For we have $QP^2 = CQ \cdot QC'$ and from the triangles CQO, $QC'O'$, observing that $\measuredangle\, QO'C' = \beta - 2\,\alpha$:

$$\frac{CQ}{OQ} = \frac{\sin\beta}{\sin(\frac{1}{2}\pi - \alpha)}, \qquad \frac{QC'}{QO'} = \frac{\sin(\beta - 2\,\alpha)}{\sin(\frac{1}{2}\pi + \alpha)},$$

whence

$$\frac{QP^2}{OQ \cdot QO'} = \frac{\sin\beta \sin(\beta - 2\alpha)}{\cos^2\alpha},$$

an expression independent of the position of the cross-section CC'.

Denoting this positive constant by k^2, we find the equation

$$y^2 = k^2x(2a - x),$$

i.e.

$$\frac{(x-a)^2}{a^2} + \frac{y^2}{(ka)^2} = 1,$$

which represents an ellipse, with semi-axes a, ka and center $(a, 0)$.

242. Hyperbola. In the third case (Fig. 100), proceeding as in the second and merely observing that now

$$QO' = -(2a + x),$$

we find the equation

$$y^2 = k^2x(2a + x),$$

i.e.

$$\frac{(x+a)^2}{a^2} - \frac{y^2}{(ka)^2} = 1,$$

which represents a hyperbola, with semi-axes a, ka and center $(-a, 0)$.

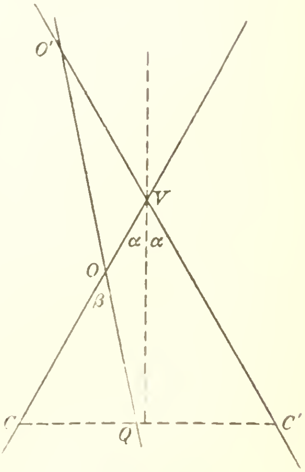


Fig. 100

243. Limiting Cases. As the conic is an ellipse, hyperbola, or parabola according as $\beta > 2\alpha$, $< 2\alpha$, or $= 2\alpha$, it appears that the *parabola* can be regarded as the limiting case of either an ellipse or a hyperbola whose center (the midpoint of OO') is removed to infinity.

On the other hand, if in the second case, $\beta > 2\alpha$ (Fig. 99), we let β approach π, or if in the third case, $\beta < 2\alpha$ (Fig. 100), we let β approach 0, the cutting plane becomes in the limit a tangent plane to the cone. It then has in common with the cone the points of the generator VC, and these only. A *single straight line* can thus appear as a limiting case of an ellipse or hyperbola.

Finally we obtain another class of limiting cases, or *cases of degeneration*, of the conics if, in any one of the three cases, we let the cutting plane pass through the vertex V of the cone. In the first case, $\beta = 2\alpha$, the cutting plane is then tangent to the cone so that the parabola also may degenerate into a single straight line. In the second case, $\beta > 2\alpha$, if $\beta \neq \pi$, the ellipse degenerates into a single point, the vertex V of the cone. In the third case, $\beta < 2\alpha$, if $\beta \neq 0$, the hyperbola degenerates into two intersecting lines.

The term conic section, or *conic*, is often used as including these limiting cases.

EXERCISES

1. For what value of β in the preceding discussion does the conic become a circle?

2. A right circular cylinder can be regarded as the limiting case of a right circular cone whose vertex is removed to infinity along its axis while a certain cross-section remains fixed. The section of such a cylinder by a plane is in general an ellipse; in what case does it degenerate into two parallel lines?

3. The conic sections were originally defined (by the older Greek mathematicians, in the time of Plato, about 400 B.C.) as sections of a cone by a plane at right angles to a generator of the cone; show that the section is a parabola, ellipse, or hyperbola according as the angle 2α at the vertex of the cone is $= \frac{1}{2}\pi$, $< \frac{1}{2}\pi$, $> \frac{1}{2}\pi$.

4. Show that the spheres inscribed in a right circular cone so as to touch the cutting plane (Figs. 98, 99, 100) touch this plane at the foci of the conic.

PART II. REDUCTION OF GENERAL EQUATION

244. Equations of Conics. We have seen in the two preceding chapters that *by selecting the coordinate system in a convenient way* the equation of a *parabola* can be obtained in the simple form

$$y^2 = 4\,px,$$

that of an *ellipse* in the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

and that of a *hyperbola* in the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

When the coordinate system is taken arbitrarily, the cartesian equations of these curves will in general not have this simple form; but they will always be of the second degree. To show this let us take the common definition of these curves (§ 232) as the locus of a point whose distances from a fixed point and a fixed line are in a constant ratio. With respect to any rectangular axes, let x_1, y_1 be the coordinates of the fixed point, $ax + by + c = 0$ the equation of the fixed line, and e the given ratio. Then by §§ 9 and 56 the equation of the locus is

$$\sqrt{(x-x_1)^2 + (y - y_1)^2} = e \cdot \frac{ax + by + c}{\pm\sqrt{a^2 + b^2}},$$

or, rationalized:

$$(x - x_1)^2 + (y - y_1)^2 = \frac{e^2}{a^2 + b^2}(ax + by + c)^2.$$

It is readily seen that this equation is always of the second degree; *i.e.* that the coefficients of x^2, y^2, and xy cannot all three vanish.

245. Equation of Second Degree. Conversely, *every equation of the second degree, i.e.* every equation of the form (§ 79)

(1) $$Ax^2 + 2Hxy + By^2 + 2Gx + 2Fy + C = 0,$$

where A, H, B are not all three zero, *in general represents a conic.* More precisely, the equation (1) may represent an ellipse, a hyperbola, or a parabola; it may represent two straight lines, different or coincident; it may be satisfied by the coordinates of only a single point; and it may not be satisfied by any real point.

Thus each of the equations

$$x^2 - 3y^2 = 0,\ xy = 0$$

evidently represent two real different lines; the equation

$$x^2 - 2x + 1 = 0$$

represents a single line, or as it is customary to say, two coincident lines; the equation

$$x^2 + y^2 = 0$$

represents a single point, while

$$x^2 + y^2 + 1 = 0$$

is satisfied by no real point and is sometimes said to represent an "imaginary ellipse."

The term *conic* is often used in a broader sense (compare § 243) so as to include all these cases; it is then equivalent to the expression "locus of an equation of the second degree."

It will be shown in the present chapter how to determine the locus of any equation of the form (1) with real coefficients. The method consists in selecting the axes of coordinates so as to reduce the given equation to its most simple form.

246. Translation of Axes. The transformation of the equation (1) to its most simple form is very easy in the particular case *when* (1) *contains no term in xy, i.e.* when $H = 0$. Indeed it suffices in this case *to complete the squares in x and y and transform to parallel axes.*

Two cases may be distinguished:

(*a*) $H = 0,\ A \neq 0,\ B \neq 0$, so that the equation has the form

$$Ax^2 + By^2 + 2\,Gx + 2\,Fy + C = 0. \tag{2}$$

Completing the squares in x and y (§ 80), we obtain an equation of the form

$$A\,(x-h)^2 + B\,(y-k)^2 = K,$$

where K is a constant; upon taking parallel axes through the point (h, k) it is seen that the locus is an ellipse, or a hyperbola, or two straight lines, or a point, or no real locus, according to the values of A, B, K.

(*b*) $H = 0$, and either $B = 0$ or $A = 0$, so that the equation is

$$Ax^2 + 2\,Gx + 2\,Fy + C = 0, \text{ or } By^2 + 2\,Gx + 2\,Fy + C = 0. \tag{3}$$

Completing the square in x or y, we obtain

$$(x-h)^2 = p\,(y-k), \text{ or } (y-k)^2 = q\,(x-h);$$

with (h, k) as new origin we have a parabola referred to vertex and axis, or two parallel lines, real and different, coincident, or imaginary.

It follows from this discussion that *the absence of the term in xy indicates that, in the case of the ellipse or hyperbola, its axes, in the case of the parabola, its axis and tangent at the vertex, are parallel to the axes of coordinates.*

EXERCISES

1. Reduce the following equations to standard forms and sketch the loci:

(*a*) $2\,y^2 - 3\,x + 8\,y + 11 = 0$,

(*b*) $x^2 + 4\,y^2 - 6\,x + 4\,y + 6 = 0$,

(*c*) $6\,x^2 + 3\,y^2 - 4\,x + 2\,y + 1 = 0$,

(*d*) $x^2 - 9\,y^2 - 6\,x + 18\,y = 0$,

(*e*) $9\,x^2 + 9\,y^2 - 36\,x + 6\,y + 10 = 0$,

(*f*) $2\,x^2 - 4\,y^2 + 4\,x + 4\,y - 1 = 0$,

(*g*) $x^2 + y^2 - 2\,x + 2\,y + 3 = 0$,

(*h*) $3\,x^2 - 6\,x + y + 6 = 0$,

(*i*) $x^2 - y^2 - 4\,x - 2\,y + 3 = 0$,

(*j*) $2\,x^2 - 5\,x + 12 = 0$,

(*k*) $2\,x^2 - 5\,x + 2 = 0$,

(*l*) $y^2 - 4\,y + 4 = 0$.

2. Find the equation of each of the following conics, determine the axis perpendicular to the given directrix, the vertices on this axis (by division-ratio), the lengths of the semi-axes, and make a rough sketch in each case:

(*a*) with $x - 2 = 0$ as directrix, focus at $(6, 3)$, eccentricity $\frac{1}{3}$;

(*b*) with $3x + 4y - 6 = 0$ as directrix, focus at $(5, 4)$, eccentricity $\frac{1}{2}$;

(*c*) with $x - y - 2 = 0$ as directrix, focus at $(4, 0)$, eccentricity $\frac{3}{2}$.

3. Find the axis, vertex, latus rectum, and sketch the parabola with focus at $(2, -2)$ and $2x - 3y - 5 = 0$ as directrix (see Ex. 2).

4. Prove the statement at the end of § 244.

5. Find the equation of the ellipse of major axis 5 with foci at $(0, 0)$ and $(3, 1)$.

247. Rotation of Axes. If the right angle xOy formed by the axes Ox, Oy be turned about the origin O through an angle θ so as to take the new position x_1Oy_1 (Fig. 101), the relation between the old coordinates $OQ = x$, $QP = y$ of any point P and the new coordinates $OQ_1 = x_1$, $Q_1P = y_1$ of the same point P are seen from the figure to be

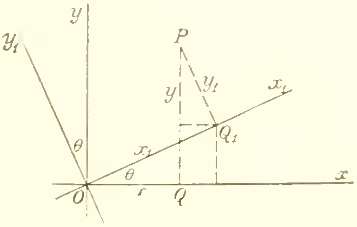


FIG. 101

$$(4) \qquad \begin{cases} x = x_1 \cos\theta - y_1 \sin\theta, \\ y = x_1 \sin\theta + y_1 \cos\theta. \end{cases}$$

By solving for x_1, y_1, or again from Fig. 101, we find

$$(4') \qquad \begin{cases} x_1 = x \cos\theta + y \sin\theta, \\ y_1 = -x \sin\theta + y \cos\theta. \end{cases}$$

If the cartesian equation of any curve referred to the axes

Ox, Oy is given, the equation of the same curve referred to the new axes Ox_1, Oy_1 is found by substituting the values (4) for x, y in the given equation.

248. Translation and Rotation. To transform from any rectangular axes Ox, Oy (Fig. 102) to any other rectangular

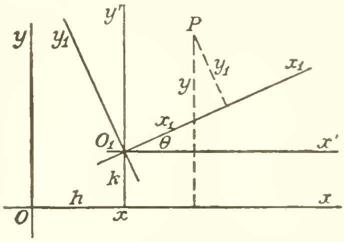


FIG. 102

axes O_1x_1, O_1y_1, we have to combine the translation OO_1 (§ 13) with the rotation through an angle θ (§ 247).

This can be done by first transforming from Ox, Oy to the parallel axes O_1x', O_1y' by means of the translation (§ 13)

$$x = x' + h,$$
$$y = y' + k,$$

and then turning the right angle $x'O_1y'$ through the angle $\theta = x'O_1x_1$, which is done by the transformation (§ 247)

$$x' = x_1 \cos\theta - y_1 \sin\theta,$$
$$y' = x_1 \sin\theta + y_1 \cos\theta.$$

Eliminating x', y', we find

$$(5) \qquad \begin{cases} x = x_1 \cos\theta - y_1 \sin\theta + h, \\ y = x_1 \sin\theta + y_1 \cos\theta + k. \end{cases}$$

The same result would have been obtained by performing first the rotation and then the translation.

It has been assumed that the right angles xOy and x_1Oy_1 are *superposable*; if this were not the case, it would be necessary to invert ultimately one of the axes.

EXERCISES

1. Find the coordinates of each of the following points after the axes have been rotated about the origin through the indicated angle:

(*a*) $(3, 4)$, $\frac{1}{4}\pi$.
(*b*) $(0, 5)$, $\frac{1}{2}\pi$.
(*c*) $(-3, 2)$, $\theta = \tan^{-1}\frac{3}{2}$.
(*d*) $(4, -3)$, $\frac{1}{3}\pi$.

2. If the origin is moved to the point $(2, -1)$ and the axes then rotated through 30°, what will be the new coordinates of the following points?

(*a*) $(0, 0)$. (*b*) $(2, 3)$. (*c*) $(6, -1)$.

3. Find the new equation of the parabola $y^2 = 4\,ax$ after the axes have been rotated through: (*a*) $\frac{1}{4}\pi$, (*b*) $\frac{1}{2}\pi$, (*c*) π.

4. Show that the equation $x^2 + y^2 = a^2$ is not changed by any rotation of the axes about the origin. Why is this true?

5. Find the center of the circle $(x - a)^2 + y^2 = a^2$ after the axes have been turned about the origin through the angle θ. What is the new equation?

6. For each of the following loci rotate the axes about the origin through the indicated angle and find the new equation:

(*a*) $x^2 - y^2 + 2 = 0$, $\frac{1}{4}\pi$.
(*b*) $x^2 - y^2 = a^2$, $\frac{1}{2}\pi$.
(*c*) $y = mx + b$, $\theta = \tan^{-1} m$.
(*d*) $12\,x^2 - 7\,xy - 12\,y^2 = 0$, $\theta = \tan^{-1}\frac{3}{4}$.
(*e*) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $\frac{1}{2}\pi$.
(*f*) $x^2 - y^2 = 0$, $\frac{1}{4}\pi$.

7. Through what angle must the axes be turned about the origin so that the circle $x^2 + y^2 - 3\,x + 4\,y - 5 = 0$ will not contain a linear term in x?

8. Suppose the right angle x_1Oy_1 (Fig. 101) turns about the origin at a uniform rate making one complete revolution in two seconds. The coordinates of a point with respect to the moving axes being $(2, 1)$, what are its coordinates with respect to the fixed axes xOy at the end of: (*a*) $\frac{1}{2}$ sec.? (*b*) $\frac{2}{3}$ sec.? (*c*) 1 sec.? (*d*) $1\frac{1}{2}$ sec.?

9. In Fig. 101, draw the line OP, and denote $\angle\,QOP$ by ϕ. Divide both sides of each of the equations (4) by OP and show that they are then equivalent to the trigonometric formulas for $\cos(\theta + \phi)$ and $\sin(\theta + \phi)$.

249. **Removal of the Term in xy.** The general equation of the second degree (1), § 245, when the axes are turned about the origin through an angle θ (§ 247), becomes:

$$\begin{aligned}&A(x_1\cos\theta - y_1\sin\theta)^2\\ &+2H(x_1\cos\theta - y_1\sin\theta)(x_1\sin\theta + y_1\cos\theta)\\ &+B(x_1\sin\theta + y_1\cos\theta)^2\\ &+2G(x_1\cos\theta - y_1\sin\theta)\\ &+2F(x_1\sin\theta + y_1\cos\theta) + C = 0.\end{aligned}$$

This is an equation of the second degree in x_1 and y_1 in which the coefficient of x_1y_1 is readily seen to be

$$\begin{aligned}-2A\cos\theta\sin\theta + 2B\sin\theta\cos\theta + 2H(\cos^2\theta - \sin^2\theta)&\\ = (B - A)\sin 2\theta + 2H\cos 2\theta.&\end{aligned}$$

It follows that if the axes be turned about the origin through an angle θ such that

$$(B - A)\sin 2\theta + 2H\cos 2\theta = 0,$$

i.e. such that

$$\tan 2\theta = \frac{2H}{A - B}, \tag{6}$$

the equation referred to the new axes will contain no term in x_1y_1 and can therefore be treated by the method of § 246. According to the remark at the end of § 246 this means that the new axes Ox_1, Oy_1, obtained by turning the original axes Ox, Oy through the angle θ found from (6), are parallel to the axes of the conic (or, in the case of the parabola, to the axis and the tangent at the vertex).

The equation (6) can therefore be used to determine *the directions of the axes of the conic;* but the process just indicated is generally inconvenient for reducing a numerical equation of the second degree to its most simple form since the values of $\cos\theta$ and $\sin\theta$ required by (4) to obtain the new equation are in general irrational.

EXERCISES

1. Through what angle must the axes be turned about the origin to remove the term in xy from each of the following equations?

(*a*) $3x^2+2\sqrt{3}xy+y^2-3x+4y-10=0.$ (*b*) $x^2+2\sqrt{3}xy+7y^2-15=0$

(*c*) $2x^2-3xy+2y^2+x-y+7=0.$ (*d*) $xy=2a^2.$

2. Reduce each of the following equations to one of the forms in § 244.

(*a*) $xy=-2.$ (*b*) $6x^2-5xy-6y^2=0.$

(*c*) $3x^2-10xy+3y^2+8=0.$ (*d*) $13x^2-10xy+13y^2-72=0.$

250. Transformation to Parallel Axes. To transform the general equation of the second degree (1), § 245, to parallel axes through any point (x_0, y_0), we have to substitute (§ 13)

$$x=x'+x_0, \quad y=y'+y_0,$$

the resulting equation is

$$Ax'^2+2Hx'y'+By'^2+2(Ax_0+Hy_0+G)x' \\ +2(Hx_0+By_0+F)y'+C'=0,$$

where the new constant term is

$$C'=Ax_0^2+2Hx_0y_0+By_0^2+2Gx_0+2Fy_0+C. \tag{7}$$

It thus appears that *after any translation of the coordinate system:*

(*a*) the coefficients of the terms of the second degree remain unchanged;

(*b*) the new coefficients of the terms of the first degree are linear functions of the coordinates of the new origin;

(*c*) the new constant term is the result of substituting the coordinates of the new origin in the left-hand member of the original equation.

251. Transformation to the Center. The transformed equation will contain no terms of the first degree, *i.e.* it will be of the form

$$(8) \qquad Ax'^2 + 2\,Hx'y' + By'^2 + C' = 0,$$

if we can select the new origin $(x_0,\ y_0)$ so that

$$(9) \qquad \begin{aligned} Ax_0 + Hy_0 + G &= 0, \\ Hx_0 + By_0 + F &= 0. \end{aligned}$$

This is certainly possible whenever

$$AB - H^2 = \begin{vmatrix} A & H \\ H & B \end{vmatrix} \neq 0,$$

and we then find:

$$(10) \qquad x_0 = \frac{FH - GB}{AB - H^2}, \quad y_0 = \frac{GH - FA}{AB - H^2}.$$

As the equation (8) remains unchanged when x', y' are replaced by $-x'$, $-y'$, respectively, the new origin so found is the **center** of the curve (§ 224). The locus is therefore in this case a *central conic*, *i.e.* an ellipse or a hyperbola; but it may reduce to two straight lines or to a point (see § 254). It might be entirely imaginary, viz. if $H = 0$; but the case when $H = 0$ has already been discussed in § 246.

We shall discuss in § 256 the case in which $AB - H^2 = 0$.

252. The Constant Term and the Discriminant. The calculation of the constant term C' can be somewhat simplified by observing that its expression (7) can be written

$$C' = (Ax_0 + Hy_0 + G)x_0 + (Hx_0 + By_0 + F)y_0 + Gx_0 + Fy_0 + C,$$

i.e., owing to (9),

$$(11) \qquad C' = Gx_0 + Fy_0 + C.$$

If we here substitute for x_0, y_0 their values (10) we find:

$$C' = \frac{GFH - G^2B + FGH - F^2A + ABC - H^2C}{AB - H^2}.$$

The numerator, which is called the ***discriminant*** of the equation of the second degree and is denoted by D, can be written in the form of a symmetric determinant, viz.

$$D = \begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix}.$$

If we denote the cofactors of this determinant by the corresponding small letters, we have

$$x_0 = \frac{g}{c},\ y_0 = \frac{f}{c},\ C' = \frac{D}{c}.$$

Notice that the coefficients of the equations (9), which determine the center, are given by the first two rows of D, while the third row gives the coefficients of C' in (11).

253. Homogeneous Function of Second Degree. The notation for the coefficients in the equation of the second degree arises from the fact that the left-hand member of this equation can be regarded as the value for $z = 1$ of the general *homogeneous* function of the second degree, viz.

$$f(x, y, z) = Ax^2 + By^2 + Cz^2 + 2\,Fyz + 2\,Gzx + 2\,Hxy.$$

If in this function x alone be regarded as variable while y and z are treated as constants, the derivative with respect to x is

$$f_x' = 2(Ax + Hy + Gz);$$

if y alone, or z alone, be regarded as variable, we find similarly

$$f_y' = 2(Hx + By + Fz),$$
$$f_z' = 2(Gx + Fy + Cz).$$

These *partial* derivatives of the homogeneous function $f(x, y, z)$ with respect to x, y, z, respectively, are linear homogeneous functions of x, y, z, and it is at once verified that

$$f = \tfrac{1}{2}(f_x' \cdot x + f_y' \cdot y + f_z' \cdot z);$$

i.e. *the homogeneous function of the second degree is equal to half the sum of the products of its partial derivatives by x, y, z.*

The left-hand members of the equations (9) are $\frac{1}{2}f_x'(x_0, y_0, 1)$, $\frac{1}{2}f_y'(x_0, y_0, 1)$. Hence *the equations for the center can be obtained by differentiating* $f(x, y, z)$, or what amounts to the same, *the left-hand member of the equation of the second degree, with respect to x alone and y alone.*

The symmetric determinant

$$D = \begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix}$$

formed of the coefficients of $\frac{1}{2}f_x'$, $\frac{1}{2}f_y'$, $\frac{1}{2}f_z'$ is called the *discriminant* of $f(x, y, z)$; and this is also the discriminant of the equation of the second degree (§ 252). As $f = \frac{1}{2}(f_x'x + f_y'y + f_z'z)$ and $f_x'(x_0, y_0, 1) = 0$, $f_y'(x_0, y_0, 1) = 0$ it follows that

$$C' = f(x_0, y_0, 1) = \tfrac{1}{2}f_z'(x_0, y_0, 1) = Gx_0 + Fy_0 + C.$$

254. Straight Lines. After transforming to the center, *i.e.* obtaining the equation (8), we must distinguish two cases according as $C' = 0$ or $C' \neq 0$. The condition $C' = 0$ means by (7) that the center lies on the locus; and indeed the homogeneous equation

$$Ax'^2 + 2\,Hx'y' + By'^2 = 0$$

represents two straight lines through the new origin (x_0, y_0) (§ 59). The separate equations of these lines, referred to the new axes, are found by factoring the left-hand member. As we here assume (§ 251) that $AB - H^2 \neq 0$, and $H \neq 0$, the lines can only be either real and different, or imaginary. In the latter case the point (x_0, y_0) is the only real point whose coordinates satisfy the original equation.

255. Ellipse and Hyperbola. If $C' \neq 0$ we can divide (8) by $-C'$ so that the equation reduces to the form

$$(12) \qquad ax^2 + 2\,hxy + by^2 = 1.$$

This equation represents an ellipse or a hyperbola (since we assume $h \neq 0$). The axes of the ellipse or hyperbola can be found in magnitude and direction as follows.

If an ellipse or hyperbola, with its center, be given graphically, the axes can be constructed by intersecting the curve with a concentric circle and drawing the lines from the center to the intersections: the bisectors of the angles between these lines are evidently the axes of the curve (Fig. 103).

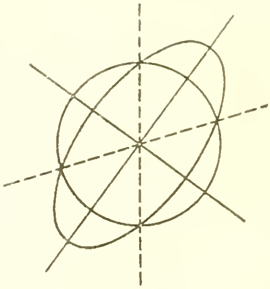


FIG. 103

The intersections of the curve (12) with a concentric circle of radius r are given by the simultaneous equations

$$ax^2 + 2\,hxy + by^2 = 1, \qquad x^2 + y^2 = r^2;$$

dividing the second equation by r^2 and subtracting it from the first, we have

$$(13) \qquad \left(a - \frac{1}{r^2}\right)x^2 + 2\,hxy + \left(b - \frac{1}{r^2}\right)y^2 = 0.$$

This *homogeneous* equation represents two straight lines through the origin, and as the equation is satisfied by the coordinates of the points that satisfy both the preceding equations, these lines must be the lines from the origin to the intersections of the circle with the curve (12). If we now select r so as to make the two lines (13) coincide, they will evidently coincide with one or the other of the axes of the curve (12). The condition for equal roots of the quadratic (13) in y/x is

$$(14) \qquad \left(a - \frac{1}{r^2}\right)\left(b - \frac{1}{r^2}\right) - h^2 = 0.$$

This equation, which is quadratic in $1/r^2$ and can be written

$$(14') \qquad \left(\frac{1}{r^2}\right)^2 - (a + b)\frac{1}{r^2} + ab - h^2 = 0,$$

determines *the lengths of the axes.* If the two values found for r^2 are both positive, the curve is an ellipse: if one is positive and the other negative, it is a hyperbola; if both are negative, there is no real locus.

Each of the two values of $1/r^2$ found from (14'), if substituted in (13), makes the left-hand member, owing to (14), a complete square. *The equations of the axes* are therefore

$$\sqrt{a - \frac{1}{r^2}}\,x \pm \sqrt{b - \frac{1}{r^2}}\,y = 0,$$

or, multiplying by $\sqrt{a - 1/r^2}$ and observing (14):

$$\left(a - \frac{1}{r^2}\right)x + hy = 0.$$

256. Parabola. It remains to discuss the case (§ 251) of the general equation of the second degree,

$$Ax^2 + 2Hxy + By^2 + 2Gx + 2Fy + C = 0,$$

in which we have $$AB - H^2 = 0.$$

This condition means that *the terms of the second degree form a perfect square:*

$$Ax^2 + 2Hxy + By^2 = (\sqrt{A}x + \sqrt{B}y)^2.$$

Putting $\sqrt{A} = a$ and $\sqrt{B} = b$ we can write the equation of the second degree in this case in the form

$$(15) \qquad (ax + by)^2 = -2Gx - 2Fy - C.$$

If G and F are both zero, this equation represents *two parallel straight lines*, real and different, real and coincident, or imaginary according as $C < 0$, $C = 0$, $C > 0$.

If G and F are not both zero, the equation (15) can be interpreted as meaning that the square of the distance of the point (x, y) from the line

$$(16) \qquad ax + by = 0$$

is proportional to the distance of (x, y) from the line

$$(17) \qquad 2Gx + 2Fy + C = 0.$$

Hence if these lines (16), (17) happen to be at right angles, the locus of (15) is a *parabola*, having the line (16) as axis and the line (17) as tangent at the vertex.

But even when the lines (16) and (17) are not at right angles the equation (15) can be shown to represent a parabola. For if we add a constant k within the parenthesis and compensate the right-hand member by adding the terms $2\,akx + 2\,bky + k^2$, the locus of (15) is not changed; and in the resulting equation

$$(18)\quad (ax + by + k)^2 = 2(ak - G)x + 2(bk - F)y + k^2 - C$$

we can determine k so as to make the two lines

$$(19)\qquad ax + by + k = 0,$$

$$(20)\qquad 2(ak - G)x + 2(bk - F)y + k^2 - C = 0$$

perpendicular. The condition for perpendicularity is

$$a(ak - G) + b(bk - F) = 0,$$

whence

$$(21)\qquad k = \frac{aG + bF}{a^2 + b^2}.$$

With this value of k, then, the lines (19), (20) are at right angles; and if (19) is taken as new axis Ox and (20) as new axis Oy, the equation (18) reduces to the simple form

$$y^2 = px.$$

The constant p, *i.e.* the latus rectum of the parabola, is found by writing (18) in the form

$$\left(\frac{ax + by + k}{\sqrt{a^2 + b^2}}\right)^2 =$$

$$\frac{2\sqrt{(ak - G)^2 + (bk - F)^2}}{a^2 + b^2} \cdot \frac{2(ak - G)x + 2(bk - F)y + k^2 - C}{2\sqrt{(ak - G)^2 + (bk - F)^2}};$$

hence

$$p = \frac{2}{a^2 + b^2}\sqrt{(ak - G)^2 + (bk - F)^2}.$$

Substituting for k its value (21) we can reduce it to

$$p = \frac{2(aF - bG)}{(a^2 + b^2)^{\frac{3}{2}}}.$$

EXERCISES

1. Find the equation of each of the following loci after transforming to parallel axes through the center:

(*a*) $3x^2 - 4xy - y^2 - 3x - 4y + 7 = 0.$
(*b*) $5x^2 + 6xy + y^2 + 6x - 4y - 5 = 0.$
(*c*) $2x^2 + xy - 6y^2 - 7x - 7y + 5 = 0.$
(*d*) $x^2 - 2xy - y^2 + 4x - 2y - 8 = 0.$

2. Find that diameter of the conic $3x^2 - 2xy - 4y^2 + 6x - 4y + 2 = 0$ (*a*) which passes through the origin, (*b*) which is parallel to each coordinate axis.

3. For what values of k do the following equations represent straight lines? Find their intersections.

(*a*) $2x^2 - xy - 3y^2 - 6x + 19y + k = 0.$
(*b*) $kx^2 + 2xy + y^2 - x - y - 6 = 0.$
(*c*) $3x^2 - 4xy + ky^2 + 8y - 3 = 0.$
(*d*) $x^2 + 2y^2 + 6x - 4y + k = 0.$

4. Show that the equations of conjugate hyperbolas $x^2/a^2 - y^2/b^2 = \pm 1$ and their asymptotes $x^2/a^2 - y^2/b^2 = 0$, even after a translation and rotation of the axes, will differ only in the constant terms and that the constant term of the asymptotes is the arithmetic mean between the constant terms of the conjugate hyperbolas.

5. Find the asymptotes and the hyperbola conjugate to

$$2x^2 - xy - 15y^2 + x + 19y + 16 = 0.$$

6. Find the hyperbola through the point $(-2, 1)$ which has the lines $2x - y + 1 = 0$, $3x + 2y - 6 = 0$ as asymptotes. Find the conjugate hyperbola.

7. Show that the hyperbola $xy = a^2$ is referred to its asymptotes as coordinate axes. Find the semi-axes and sketch the curve. Find and sketch the conjugate hyperbola.

8. The volume of a gas under constant temperature varies inversely as the pressure (Boyle's law), *i.e.* $vp = c$. Sketch the curve whose ordinates represent the pressure as a function of the volume for different values of c; *e.g.* take $c = 1, 2, 3$.

9. Sketch the hyperbola $(x - a)(y - b) = c^2$ and its asymptotes. Interpret the constants a, b, c geometrically.

10. Sketch the hyperbola $xy + 3y - 6 = 0$ and its asymptotes.

11. Find the center and semi-axes of the following conics, write their equations in the most simple form, and sketch the curves:

(*a*) $5x^2 - 6xy + 5y^2 + 12\sqrt{2}x - 4\sqrt{2}y + 8 = 0$.

(*b*) $x^2 - 6\sqrt{3}xy - 5y^2 - 16 = 0$. (*c*) $x^2 + xy + y^2 - 3y + 6 = 0$.

✓ (*d*) $13x^2 - 6\sqrt{3}xy + 7y^2 - 64 = 0$.

→ (*e*) $2x^2 - 4xy + y^2 + 2x - 4y - \frac{5}{2} = 0$.

(*f*) $3x^2 + 2xy + y^2 + 6x + 4y + \frac{7}{2} = 0$.

12. Sketch the following parabolas:

(*a*) $x^2 - 2\sqrt{3}xy + 3y^2 - 6\sqrt{3}x - 6y = 0$.

(*b*) $x^2 - 6xy + 9y^2 - 3x + 4y - 1 = 0$.

13. Show that the following combinations of the coefficients of the general equation of the second degree are *invariants* (*i.e.* remain unchanged) under any transformation from rectangular to rectangular axes:

(*a*) $A + B$. (*b*) $AB - H^2$. (*c*) $(A - B)^2 + 4H^2$.

14. Show that $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$ represents a parabola. Sketch the locus.

15. Find the parabola with $x + y = 0$ as directrix and $(\frac{1}{2}a, \frac{1}{2}a)$ as focus.

16. Let five points A, B, C, D, E be taken at equal intervals on a line. Show that the locus of a point P such that $AP \cdot EP = BP \cdot DP$ is an equilateral hyperbola. (Take C as origin.)

17. The variable triangle AQB is isosceles with a fixed base AB. Show that the locus of the intersection of the line AQ with the perpendicular to QB through B is an equilateral hyperbola.

18. Let A be a fixed point and let Q describe a fixed line. Find the locus of the intersection of a line through Q perpendicular to the fixed line and a line through A perpendicular to AQ.

19. Find the locus of the intersection of lines drawn from the extremities of a fixed diameter of a circle to the ends of the perpendicular chords.

20. Show by (14′), § 255, that if the equation of the second degree represents an ellipse, parabola, hyperbola, we have, respectively,

$$AB - H^2 > 0, = 0, < 0.$$

CHAPTER XII

HIGHER PLANE CURVES

PART I. ALGEBRAIC CURVES

257. Cubics. It has been shown (§ 30) that every equation of the first degree,

$$\begin{gathered} a_0 \\ + a_1x + b_1y = 0, \end{gathered}$$

represents a *straight line;* and (§ 245) that every equation of the second degree,

$$\begin{gathered} a_0 \\ + a_1x + b_1y \\ + a_2x^2 + b_2xy + c_2y^2 = 0, \end{gathered}$$

either represents a *conic* or is not satisfied by any real points.

The *locus* represented by an equation of the third degree,

$$\begin{gathered} a_0 \\ + a_1x + b_1y \\ + a_2x^2 + b_2xy + c_2y^2 \\ + a_3x^3 + b_3x^2y + c_3xy^2 + d_3y^3 = 0, \end{gathered}$$

i.e. the aggregate of all real points whose coordinates x, y satisfy this equation, is called a ***cubic curve.***

Similarly, the locus of all points that satisfy any equation of the *fourth* degree is called a *quartic curve;* and the terms *quintic, sextic,* etc., are applied to curves whose equations are of the *fifth, sixth,* etc., degrees.

Even the cubics present a large variety of shapes; still more so is this true of higher curves. We shall not discuss such curves in detail, but we shall study some of their properties.

258. Algebraic Curves. The general form of an *algebraic equation of the nth degree in x and y* is

$$
(1) \qquad \begin{array}{c}
a_0 \\
+ a_1x + b_1y \\
+ a_2x^2 + b_2xy + c_2y^2 \\
+ a_3x^3 + b_3x^2y + c_3xy^2 + d_3y^3 \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
+ a_nx^n + b_nx^{n-1}y + \cdots + k_nxy^{n-1} + l_ny^n = 0.
\end{array}
$$

The coefficients are supposed to be any real numbers, those in the last line being not all zero. The number of terms is not more than $1 + 2 + 3 + \cdots + (n + 1) = \frac{1}{2}(n + 1)(n + 2)$.

If the cartesian equation of a curve can be reduced to this form by rationalizing and clearing of fractions, the curve is called an *algebraic curve of degree n.*

An algebraic curve of degree n can be intersected by a straight line,

$$Ax + By + C = 0,$$

in not more than n points. For, the substitution in (1) of the value of y (or of x) derived from the linear equation gives an equation in x (or in y) of a degree not greater than n; this equation can therefore have not more than n roots, and these roots are the abscissas (or ordinates) of the points of intersection.

We have already studied the curves that represent the polynomial function

$$y = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n;$$

such a curve is an algebraic curve, but it is readily seen by comparison with the preceding equation that this equation is of a very special type, since it contains no term of higher degree than one in y. Such a curve is often called a *parabolic curve of the nth degree.*

259. Transformation to Polar Coordinates. The cartesian equation (1) is readily transformed to polar coordinates by substituting

$$x = r\cos\phi, \quad y = r\sin\phi;$$

it then assumes the form:

$$\begin{aligned}
&a_0\\
&+ (a_1\cos\phi + b_1\sin\phi)r\\
(2)\qquad &+ (a_2\cos^2\phi + b_2\cos\phi\sin\phi + c_2\sin^2\phi)r^2\\
&+ (a_3\cos^3\phi + b_3\cos^2\phi\sin\phi + c_3\cos\phi\sin^2\phi + d_3\sin^3\phi)r^3\\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\
&+ (a_n\cos^n\phi + b_n\cos^{n-1}\phi\sin\phi + \cdots + k_n\cos\phi\sin^{n-1}\phi + l_n\sin^n\phi)r^n\\
&= 0.
\end{aligned}$$

If any particular value be assigned to the polar angle ϕ, this becomes an equation in r of a degree not greater than n. Its roots $r_1, r_2, \cdots$ represent the intercepts $OP_1, OP_2, \cdots$ (Fig. 104) made by the curve (2) on the line $y = \tan\phi \cdot x$. Some of these roots may of course be imaginary, and there may be equal roots.

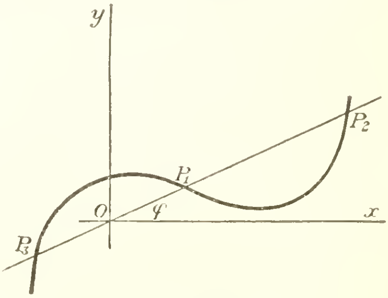


FIG. 104

260. Curve through the Origin. The equation in r has at least one of its roots equal to zero if, and only if, the constant term a_0 is zero. Thus, *the necessary and sufficient condition that the origin O be a point of the curve is* $a_0 = 0$.

This is of course also apparent from the equation (1) which is satisfied by $x = 0$, $y = 0$ if, and only if, $a_0 = 0$.

261. Tangent Line at Origin. The equation (2) has at least two of its roots equal to zero if $a_0 = 0$ and $a_1\cos\phi + b_1\sin\phi = 0$. If a_1 and b_1 are not both zero, the latter condition can be satisfied by selecting the angle ϕ properly, viz. so that

$$\tan\phi = -\frac{a_1}{b_1}.$$

The line through the origin inclined at this angle ϕ to the polar axis is the *tangent to the curve at the origin* O (Fig. 105). Its cartesian equation is $y = \tan\phi \cdot x = -(a_1/b_1)x$, *i.e.*

$$(3) \qquad a_1x + b_1y = 0.$$

Thus, if $a_0 = 0$ while a_1, b_1 are not both zero, the curve has at the origin a single tangent; the origin O is therefore called a ***simple***, or ***ordinary***, ***point*** of the curve. In other words, *if the lowest terms in the equation* (1) *of an algebraic curve are of the first degree, the origin is a simple point of the curve, and the equation of the tangent at the origin is obtained by equating to zero the terms of the first degree.*

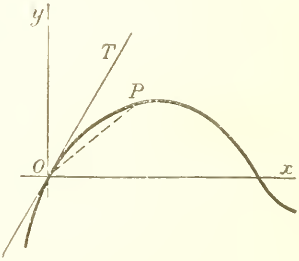


FIG. 105

262. Double Point. The condition $a_1\cos\phi + b_1\sin\phi = 0$ necessary for two zero roots is also satisfied if $a_1 = 0$ and $b_1 = 0$; indeed, it is then satisfied whatever the value of ϕ. Hence, if $a_0 = 0$, $a_1 = 0$, $b_1 = 0$, the equation (2) has at least two zero roots for any value of ϕ. If in this case the terms of the second degree in (1) do not all vanish, the curve is said to have a ***double point*** at the origin. Thus, *the origin is a double point if, and only if, the lowest terms in the equation* (1) *are of the second degree.*

263. Tangents at a Double Point. The equation (2) will have at least three of its roots equal to zero if we have $a_0 = 0$, $a_1 = 0$, $b_1 = 0$ and

$$a_2\cos^2\phi + b_2\cos\phi\sin\phi + c_2\sin^2\phi = 0.$$

If a_2, b_2, c_2 are not all zero, we can find two angles satisfying this equation which may be real and different, or real and equal, or imaginary. The lines drawn at these angles (if real) through the origin are the *tangents at the double point.*

Multiplying the last equation by r^2 and reintroducing cartesian coordinates we obtain for these tangents the equation

$$(4) \qquad a_2x^2 + b_2xy + c_2y^2 = 0.$$

Thus, *if the lowest terms in the equation* (1) *are of the second degree, the origin is a double point, and these terms of the second degree equated to zero represent the tangents at the origin.*

264. Types of Double Point. (*a*) If the two lines (4) are real and different, the double point is called a ***node*** or *crunode;* the curve then has two branches passing through the origin, each with a different tangent (Fig. 106).

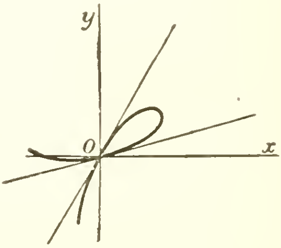


FIG. 106

(*b*) If the lines (4) are coincident, *i.e.* if $a_2x^2 + b_2xy + c_2y^2$ is a complete square, the double point is called a ***cusp***, or *spinode;* the curve then has ordinarily two real branches tangent to one and the same line at the origin (Fig. 107 represents the most simple case).

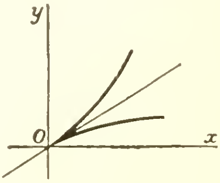


FIG. 107

(*c*) If the lines (4) are imaginary, the double point is called an ***isolated point***, or an *acnode;* in this case, while the coordinates 0, 0 of the origin satisfy the equation of the curve, there exists about the origin a region containing no other point of the curve, so that no tangents can be drawn through the origin (Fig. 108).

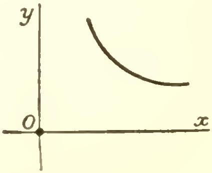


FIG. 108

It should be observed that, for curves of a degree above the third, the origin in case (*b*) may be an isolated point; this will be revealed by investigating the higher terms (viz. those above the second degree).

265. Multiple Points. It is readily seen how the reasoning of the last articles can be continued although the investigation of higher multiple points would require further discussion. The result is this: *If in the equation of an algebraic curve, when rationalized and cleared of fractions, the lowest terms are of degree k, the origin is a k-tuple point of the curve, and the tangents at this point are given by the terms of degree k, equated to zero.*

To investigate whether any given point (x_1, y_1) of an algebraic curve is simple or multiple it is only necessary to transfer the origin to the point, by replacing x by $x + x_1$ and y by $y + y_1$, and then to apply this rule.

EXERCISES

1. Determine the nature of the origin and sketch the curves:

(*a*) $y = x^2 - 2x$. (*b*) $x^2 = 4y - y^2$. (*c*) $(x + a)(y + a) = a^2$.
(*d*) $y^2 = x^2(4 - x)$. (*e*) $y^2 = x^3$. (*f*) $x^2 + y^2 = x^3$.
(*g*) $y^2 = x^2 + x^3$. (*h*) $x^3 - 3axy + y^3 = 0$. (*i*) $x^4 - y^4 + 6xy^2 = 0$.

2. Determine the nature of the origin and sketch the curve $(y - x^2)^2 = x^n$, for: (*a*) $n = 1$. (*b*) $n = 2$. (*c*) $n = 3$. (*d*) $n = 4$.

3. Locate the multiple points, determine their nature, and sketch the curves:

(*a*) $y^2 = x(x + 3)^2$. (*b*) $(y - 3)^2 = x^2$. (*c*) $(y + 1)^2 = (x - 3)^3$.
(*d*) $y^3 = (x + 1)(x - 1)^2$.

4. Sketch the curve $y^2 = (x - a)(x - b)(x - c)$ and discuss the multiple points when:

(*a*) $0 < a < b < c$. (*b*) $0 < a < b = c$. (*c*) $0 < a = b < c$. (*d*) $0 < a = b = c$.

PART II. SPECIAL CURVES
DEFINED GEOMETRICALLY OR KINEMATICALLY

266. Conchoid. *A fixed point O and a fixed line l, at the distance a from O, being given, the radius vector OQ, drawn from O to every point Q of l, is produced by a segment QP = b of constant length;* the locus of P is called the ***conchoid of Nicomedes.***

For O as pole and the perpendicular to l as polar axis the equation of l is $r_1 = a/\cos\phi$; hence that of the conchoid is

$$r = \frac{a}{\cos\phi} + b.$$

If the segment QP be laid off in the opposite sense we obtain the curve

$$r = \frac{a}{\cos\phi} - b$$

which is also called a conchoid. Indeed, these two curves are often regarded as merely two branches of the same curve. Transforming to cartesian coordinates and rationalizing, we find the equation

$$(x-a)^2(x^2+y^2) = b^2x^2,$$

which represents both branches. Sketch the curve, say for $b = 2a$, and for $b = a/2$, and determine the nature of the origin.

267. Limaçon. *If the line l be replaced by a circle and the fixed point O be taken on the circle,* the locus of P is called ***Pascal's limaçon.***

For O as pole and the diameter of the circle as polar axis the equation of the circle, of radius a, is $r_1 = 2a\cos\phi$; hence that of the limaçon is:

$$r = 2a\cos\phi + b.$$

If $b = 2a$ the curve is called the ***cardioid;*** the equation then becomes

$$r = 4a \cos^2 \tfrac{1}{2} \phi.$$

Sketch the limaçons for $b = 3a, 2a, a$; transform to cartesian coordinates and determine the character of the origin.

268. Cissoid. *$OO' = a$ being a diameter of a circle, let any radius vector drawn from O meet the circle and its tangent at O' at the points Q, D, respectively;* if on this radius vector we lay off $OR = QD$, the locus of R is called the ***cissoid of Diocles.***

With O as pole and OO' as polar axis, we have

$$OD = a/\cos\phi,\ OQ = a\cos\phi;$$

the equation is therefore

$$r = a\left(\frac{1}{\cos\phi} - \cos\phi\right) = a\frac{\sin^2\phi}{\cos\phi},$$

or in cartesian coordinates

$$y^2 = \frac{x^3}{a - x}.$$

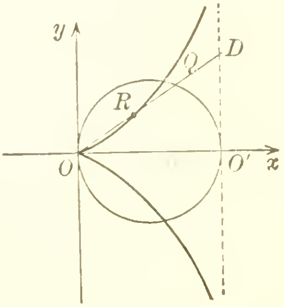


FIG. 109

If instead of taking the *difference* of the radii vectores of the circle and its tangent, we take their *sum* we obtain the so-called *companion of the cissoid,*

$$r = a(\cos\phi + \sec\phi),$$

i.e.

$$y^2 = x^2\frac{2a - x}{x - a}.$$

Sketch this curve.

269. Versiera. With the data of § 268, *let us draw through Q a parallel to the tangent, through D a parallel to the diameter;* the locus of the point of intersection P of these parallels is called the ***versiera*** (wrongly called the "witch of Agnesi").

We have evidently with O as origin and OO' as axis Ox:

$$x = a\cos^2\phi, \qquad y = a\tan\phi,$$

whence eliminating ϕ:

$$x = \frac{a^3}{y^2 + a^2}.$$

If we replace the tangent at O' by any perpendicular to OO' (Fig. 110), at the distance b from O, we obtain the curve

$$x = a\cos^2\phi, \qquad y = b\tan\phi,$$

i.e.

$$x = \frac{ab^2}{y^2 + b^2},$$

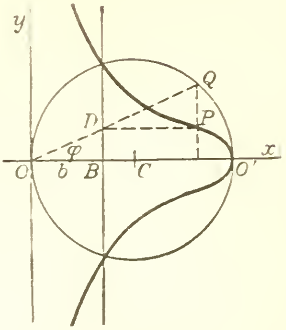


FIG. 110

which reduces to the versiera for $b = a$.

Sketch the versiera, and the last curve for $b = \frac{1}{2}a$.

270. Cassinian Ovals. Lemniscate. Two fixed points F_1, F_2 being given it is known that the locus of a point P is:

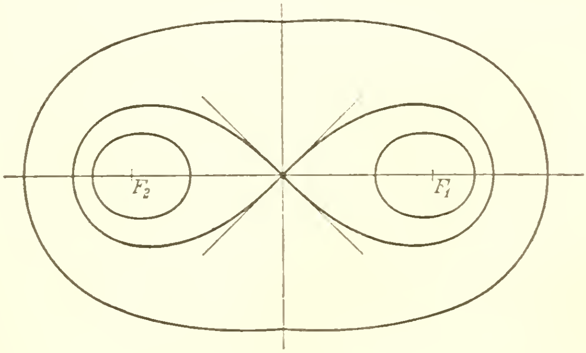


FIG. 111

(*a*) a *circle* if $F_1P/F_2P = \text{const.}$ (Ex. 7, p. 90);
(*b*) an *ellipse* if $F_1P + F_2P = \text{const.}$ (§ 204);
(*c*) a *hyperbola* if $F_1P - F_2P = \text{const.}$ (§ 207).

The locus is called a ***Cassinian oval*** if $F_1P \cdot F_2P = \text{const.}$ If we put $F_1F_2 = 2a$, the equation, referred to the midpoint O between F_1 and F_2 as origin and OF_2 as axis Ox, is

$$[(x+a)^2+y^2][(x-a)^2+y^2]=k^2.$$

In the particular case when $k = a^2$ the curve passes through the origin and is called a ***lemniscate***. The equation then reduces to the form

$$(x^2+y^2)^2 = 2a^2(x^2-y^2),$$

which becomes in polar coordinates

$$r^2 = 2a^2 \cos 2\phi.$$

Trace the lemniscate from the last equation.

271. Cycloid. The ***common cycloid*** is *the path described by any point P of a circle rolling over a straight line* (Fig. 112).

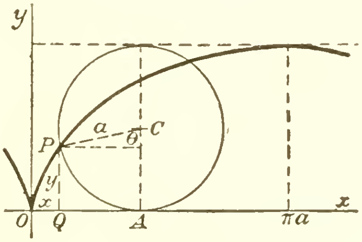


FIG. 112

If A be the point of contact of the rolling circle in any position, O the point of the given line that coincided with the point P of the circle when P was point of contact, it is clear that the length OA must equal the arc $AP = a\theta$, where a is the radius of the circle, and $\theta = \measuredangle ACP$ the angle through which the circle has turned since P was at O. The figure then shows that, with O as origin and OA as axis Ox:

$$x = OQ = a\theta - a\sin\theta, \quad y = a - a\cos\theta.$$

These are the *parameter equations* of the cycloid. The curve has an infinite number of equal arches, each with an axis of symmetry (in Fig. 112, the line $x = \pi a$) and with a cusp at each end. Write down the cartesian equation.

272. Trochoid. The path described by any point P rigidly connected with the rolling circle is called a ***trochoid.*** If the distance of P from the center C of the circle is b, the equations of the trochoid are

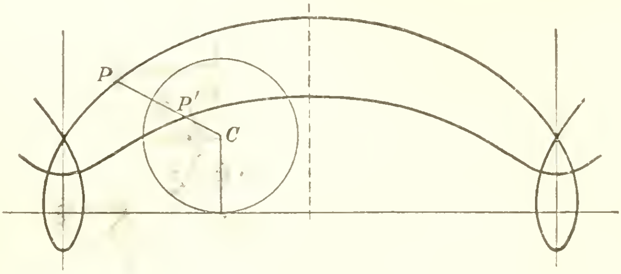


FIG. 113. — The Trochoids

$$x = a\theta - b \sin \theta, \quad y = a - b \cos \theta.$$

Draw the trochoid for $b = \frac{1}{3} a$ and for $b = \frac{4}{3} a$.

273. Epicycloid. The path described by any point P of a circle rolling on the outside of a fixed circle is called an ***epicycloid*** (Fig. 114).

Let O be the center, b the radius, of the fixed circle, C the center, a the radius, of the rolling circle; and let A_0 be that point of the fixed circle at which the describing point P is the point of contact. Put $A_0OA = \phi$, $ACP = \theta$. As the arcs AA_0 and AP are equal, we have

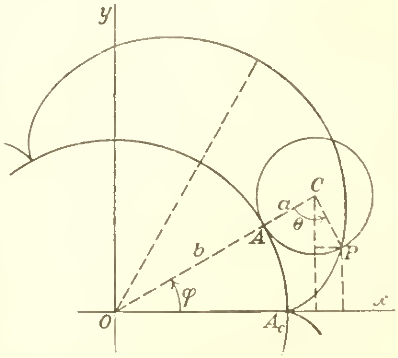


FIG. 114

$$b\phi = a\theta.$$

With O as origin and OA_0 as axis of x we have

$$x=(a+b)\cos\phi+a\sin\left[\theta-\left(\tfrac{1}{2}\pi-\phi\right)\right],$$
$$y=(a+b)\sin\phi-a\cos\left[\theta-\left(\tfrac{1}{2}\pi-\phi\right)\right],$$

i.e.
$$x=(a+b)\cos\phi-a\cos\frac{a+b}{a}\phi,$$
$$y=(a+b)\sin\phi-a\sin\frac{a+b}{a}\phi.$$

274. Hypocycloid. If the circle rolls on the inside of the fixed circle, the path of any point of the rolling circle is called a ***hypocycloid***. The equations are obtained in the same way; they differ from those of the epicycloid merely in having a replaced by $-a$:

$$x=(b-a)\cos\phi+a\cos\frac{b-a}{a}\phi.$$
$$y=(b-a)\sin\phi-a\sin\frac{b-a}{a}\phi.$$

Show that: (*a*) for $b=2a$ the hypocycloid reduces to a straight line, and illustrate this graphically; (*b*) for $b=4a$ the equations become

$$x=3a\cos\phi+a\cos 3\phi=a\cos^3\phi,$$
$$y=3a\sin\phi-a\sin 3\phi=a\sin^3\phi,$$

whence
$$x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}};$$

sketch this *four-cusped hypocycloid*.

EXERCISES

1. Sketch the following curves: (*a*) Spiral of Archimedes $r=a\phi$; (*b*) Hyperbolic spiral $r\phi=a$; (*c*) Lituus $r^2\phi=a^2$.

2. Sketch the following curves: (*a*) $r=a\sin\phi$; (*b*) $r=a\cos\phi$; (*c*) $r=a\sin 2\phi$; (*d*) $r=a\cos 2\phi$; (*e*) $r=a\cos 3\phi$; (*f*) $r=a\sin 3\phi$; (*g*) $r=a\cos 4\phi$; (*h*) $r=a\sin 4\phi$.

3. Sketch with respect to the same axes the Cassinian ovals (§ 270) for $a=1$ and $k=2, 1.5, 1.1, 1, .75, .5, 0$.

4. Let two perpendicular lines AB and CD intersect at O. Through a fixed point Q of AB draw any line intersecting CD at R. On this line lay off in both directions from R segments RP of length OR. The locus of P is called the *strophoid*. Find the equation, determine the nature of O and Q, and sketch the curve.

5. Show that the lemniscate (§ 270) is the inverse curve of an equilateral hyperbola with respect to a circle about its center.

6. Show that the strophoid (Ex. 4) is the curve inverse to an equilateral hyperbola with respect to a circle about a vertex with radius equal to the transverse axis.

7. Show that the cissoid (§ 268) is the curve inverse to a parabola with respect to a circle about its vertex.

8. Find the curve inverse to the cardioid (§ 267) with respect to a circle about the origin.

9. Transform the equation

$$a(x^2 + y^2) = x^3$$

to polar coordinates, indicate a geometrical construction, and draw the curve.

10. A tangent to a circle of radius $2a$ about the origin intersects the axes at T and T', find and sketch the locus of the midpoint P between T and T'.

11. From any point Q of the line $x = a$ draw a line parallel to the axis Ox intersecting the axis Oy at C. Find and sketch the locus of the foot of the perpendicular from C on OQ.

12. The center of a circle of radius a moves along the axis Ox. Find and sketch the locus of the intersections of this circle with lines joining the origin to its highest point.

13. The center of a circle of radius a moves along the axis Ox. Find and sketch the locus of its points of contact with the lines through the origin.

14. The center of a circle of radius a moves along the axis Ox. Its intersection with the axis nearer the origin is taken as the center of another circle which passes through the origin. Find and sketch the locus of the intersections of these circles.

PART III. SPECIAL TRANSCENDENTAL CURVES

275. The Sine Curve. The simple ***sine curve***, $y = \sin x$, is best constructed by means of an auxiliary circle of radius one. In Fig. 115, OQ is made equal to the length of the arc $OA = x$; the ordinate at Q is then equal to the ordinate BA of the circle.

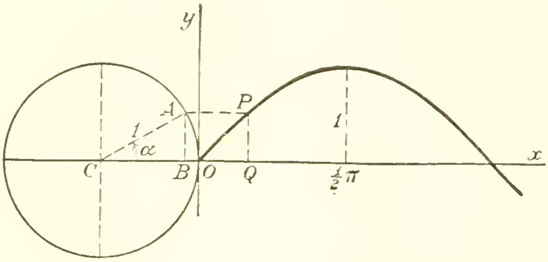


Fig. 115

Construct one whole *period* of the sine curve, *i.e.* the portion corresponding to the whole circumference of the auxiliary circle; the width 2π of this portion is called the period of the function $\sin x$.

The simple ***cosine curve***, $y = \cos x$, is the same as the sine curve except that the origin is taken at the point $(\frac{1}{2}\pi, 0)$.

The simple ***tangent curve***, $y = \tan x$, is derived like the sine curve from a unit circle. Its *period* is π.

276. The Inverse Trigonometric Curves. The equation $y = \sin x$ can also be written in the form

$$x = \sin^{-1} y, \quad \text{or } x = \text{arc} \sin y.$$

The curve represented by this equation is of course the same as that represented by the equation $y = \sin x$.

But if x and y be interchanged, the resulting equation

$$x = \sin y, \text{ or } y = \sin^{-1} x, \ y = \text{arc} \sin x,$$

represents the curve obtained from the simple sine curve by reflection in the line $y = x$ (§ 135).

Notice that the *trigonometric functions* $\sin x$, $\cos x$, $\tan x$, etc., are *one-valued, i.e.* to every value of x belongs only one value of the function, while the *inverse trigonometric functions* $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$, etc., are *many-valued*; indeed, to every value of x, at least in a certain interval, belongs an infinite number of values of the function.

EXERCISES

1. From a table of trigonometric functions, plot the curve $y = \sin x$.

2. Plot the curve $y = \sin x$ by means of the geometric construction of § 275.

3 Plot the curve $y = \cos x$ (*a*) from a table; (*b*) by a geometric construction similar to that of § 275.

4. Plot the curve $y = \tan x$ from a table.

5. Plot each of the curves

(*a*) $y = \sin 2x$.
(*b*) $y = 2 \cos 3x$.
(*c*) $y = 3 \tan (x/2)$.
(*d*) $y = \sec x$.
(*e*) $y = \text{ctn}\, 2x$.
(*f*) $y = 2 \tan 4x$.

6. Plot each of the curves

(*a*) $y = \sin^{-1} x$.
(*b*) $y = \cos^{-1} x$.
(*c*) $y = \tan^{-1} x$.

7. By adding the ordinates of the two curves $y = \sin x$ and $y = \cos x$, construct the graph of $y = \sin x + \cos x$.

8. Draw each of the curves

(*a*) $y = \sin x + 2 \cos x$.
(*b*) $y = 2 \sin x + \cos(x/2)$.
(*c*) $y = \sec x + \tan x$.
(*d*) $y = \sin x + 2 \sin 2x + 3 \sin 3x$.

9. The equation $x = \sin t$, where t means the time and x means the distance of a body from its central position, represents a *Simple Harmonic Motion*. From the graph of this equation, describe the nature of the motion.

277. Transcendental Curves. The trigonometric and inverse trigonometric curves, as well as, in general, the cycloids and trochoids, are ***transcendental curves***, so called because the relation between the cartesian coordinates x, y cannot be expressed in finite form (*i.e.* without using infinite series) by means of the *algebraic* operations of addition, subtraction, multiplication, division, and raising to a power with a constant exponent.

278. Logarithmic and Exponential Curves. Another very important transcendental curve is the ***exponential curve***

$$y = a^x,$$

and its inverse, the ***logarithmic curve***

$$y = \log_a x,$$

where a is any positive constant (Fig. 116). A full discussion of these curves can only be given in the calculus. We must here confine ourselves to some special cases and to a brief review of the fundamental laws of logarithms.

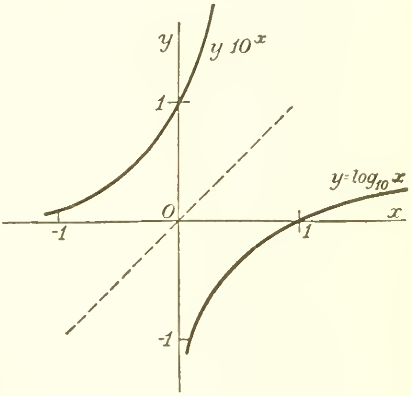


FIG. 116

279. Definitions. The ***logarithm*** b of a number c, to the base a (positive and $\neq 1$), is defined as the exponent b to which the given base a must be raised to produce the number c (§ 105); thus the two equations

$$a^b = c \text{ and } b = \log_a c$$

express exactly the same relation between b and c. It follows that

$$a^{\log_a c} = c, \text{ whatever } c.$$

If in the first law of exponents (§ 104), $a^p a^q = a^{p+q}$, we put $a^p = P$, $a^q = Q$, $a^{p+q} = N$, so that $PQ = N$, we find since $p = \log_a P$, $q = \log_a Q$, $p + q = \log_a N = \log_a PQ$:

$$(1) \qquad \log_a PQ = \log_a P + \log_a Q.$$

Similarly we find from $a^p / a^q = a^{p-q}$:

$$(2) \qquad \log_a \frac{P}{Q} = \log_a P - \log_a Q.$$

If in the third law of exponents (§ 104), $(a^p)^n = a^{pn}$, we put $a^p = P$, $a^{pn} = M$, so that $P^n = M$, we find since $p = \log_a P$, $pn = \log_a M$:

$$(3) \qquad \log_a (P^n) = n \log_a P.$$

These laws (1), (2), (3) of logarithms are merely the translation into the language of logarithms of the first and third laws of exponents.

280. Napierian or Natural Logarithms. In the ordinary tables of logarithms the base is 10, and for numerical calculations these *common logarithms* (Briggs' logarithms) are most convenient. In the calculus it is found that another system of logarithms is better adapted to theoretical considerations; the base of this system is an irrational number denoted by e,

$$e = 2.71828\,1828 \cdots,$$

and the logarithms in this system are called *natural logarithms* (or Napierian, or hyperbolic, logarithms).

281. Change of Base. Modulus. To *pass from one system of logarithms to another* observe that if the same number N has the logarithm p in the system to the base a and the logarithm q in the system to the base b so that

$$a^p = N, \quad p = \log_a N, \quad b^q = N, \quad q = \log_b N,$$

then

$$q = \log_b N = \log_b a^p = p \log_b a,$$

by (3); *i.e.*

$$(4) \qquad \log_b N = \log_a N \cdot \log_b a.$$

Hence if the logarithms of the system with the base a are known, those with the base b are found by multiplying the logarithms to the base a by a constant number, $\log_b a$.

Thus taking $a = 10$, $b = e$, we have

$$(4') \qquad \log_e N = \log_{10} N \cdot \log_e 10;$$

i.e. *to find the natural logarithm of any number we have merely to multiply its common logarithm by the number*

$$\log_e 10 = 2.30258\,509 \cdots.$$

The reciprocal of this number,

$$M = \frac{1}{\log_e 10} = 0.43429\,448 \cdots,$$

i.e. the factor by which the natural logarithms must be multiplied to produce the common logarithms, is called the ***modulus*** of the common system of logarithms.

In any system of logarithms, the logarithm of the base is always equal to 1, by the definition of the logarithm (§ 279). Hence, if in (4) we take $N = b$, we find

$$\log_a b \cdot \log_b a = 1. \tag{5}$$

In particular, with $a = 10$, $b = e$ we have

$$\log_{10} e \cdot \log_e 10 = 1\,; \tag{5'}$$

i.e. the modulus M of the common logarithms is

$$M = \frac{1}{\log_e 10} = \log_{10} e = 0.43429\,448 \cdots\ .$$

EXERCISES

1. From a table of logarithms of numbers, draw the curve $y = \log_{10} x$.

2. By multiplying the ordinates of the curve of Ex. 1 by 3, construct the curve $y = 3 \log_{10} x$.

3. From the figure of Ex. 1, construct the curve $y = 10^x$ by reflection of the curve of Ex. 1 in the line $y = x$.

4. Draw the curve $y = \frac{1}{2} \log_{10} x$ by the process of Ex. 2. Show that it represents the equation $y = \log_{100} x$, since

$$y = \log_{100} x = \log_{100} 10 \times \log_{10} x = \tfrac{1}{2} \log_{10} x.$$

5. Find $\log_{10} 7$ from a table. Construct the curve

$$y = \log_7 x = \log_{10} x \div \log_{10} 7$$

by the process described in Ex. 2 and Ex. 4.

6. Given $\log_{10} e = M = .43+$, draw the curve

$$y = \log_e x = \log_{10} x \div \log_{10} e.$$

PART IV. EMPIRICAL EQUATIONS

282. Empirical Formulas. In scientific studies, the relations between quantities are usually not known in advance, but are to be found, if possible, from pairs of numerical values of the quantities discovered by experiment.

Simple cases of this kind have already been given in §§ 15, 29. In particular, the values of a and b in formulas of the type $y = a + bx$ were found from two pairs of values of x and y. Compare also § 34.

Likewise, if two quantities y and x are known to be connected by a relation of the form $y = a + bx + cx^2$, the values of a, b, c can be found from any *three* pairs of values of x and y. For, if any pair of values of x and y are substituted for x and y in this equation, we obtain a linear equation for a, b, and c. Three such equations usually determine a, b, and c.

In general the coefficients a, b, c, $\cdots$, l in an equation of the type

$$y = a + bx + cx^2 + \cdots + lx^n$$

can be found from any $n + 1$ pairs of values of x and y.

283. Approximate Nature of Results. Since the measurements made in any experiment are liable to at least small errors, it is not to be expected that the calculated values of such coefficients as a, b, c, $\cdots$ of § 282 will be absolutely accurate, nor that the points that represent the pairs of values of x and y will all lie absolutely on the curve represented by the final formula.

To increase the accuracy, a large number of pairs of values of x and y are usually measured experimentally, and various pairs are used to determine such constants as a, b, c, $\cdots$of § 282. The *average* of all the computed values of any one such constant is often taken as a fair approximation to its true value.

284. Illustrative Examples.

Example 1. A wire under tension is found by experiment to stretch an amount l, in thousandths of an inch, under a tension T, in pounds, as follows: —

T in pounds	10	15	20	25	30
l in thousandths of an inch .	8	12.5	15.5	20	23

Find a relation of the form $l = kT$ (*Hooke's Law*) which approximately represents these results.

First plot the given data on squared paper, as in the adjoining figure.

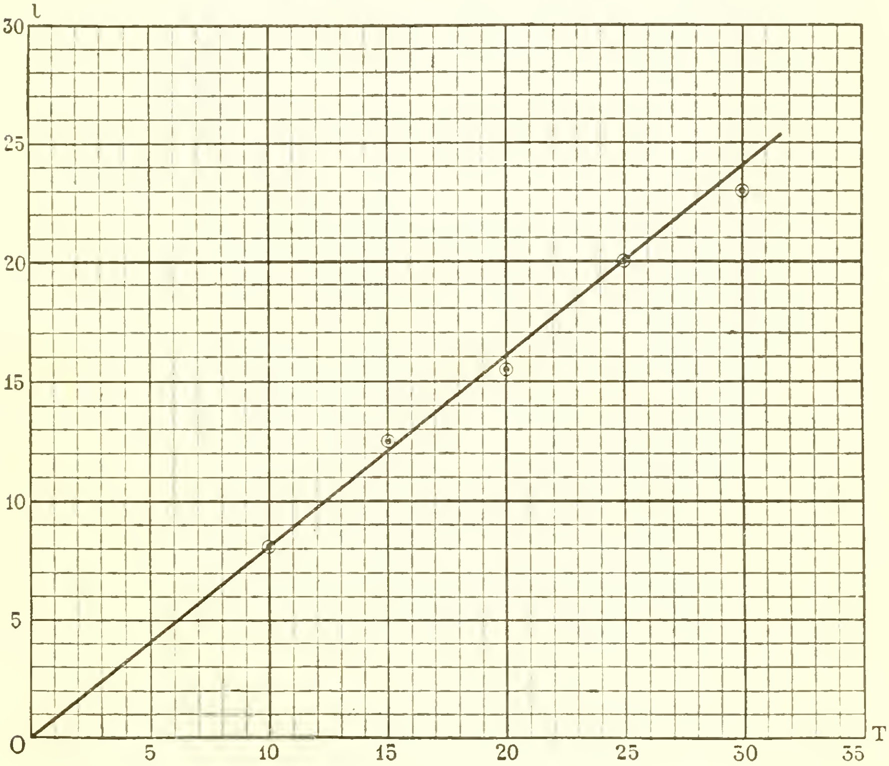


Fig. 117

Substituting $l = 8$, $T = 10$ in $l = kT$, we find $k = .8$. From $l = 12.5$, $T = 15$, we find $k = .833$. Likewise, the other pairs of values of l and T give, respectively, $k = .775$, $k = .8$, $k = .767$. The average of all these values of k is $k = .795$; hence we may write, approximately,

$$l = .795\ T.$$

This equation is represented by the line in Fig. 117; this line does not pass through even one of the given points, but it is a fair compromise between all of them, in view of the fact that each of them is itself probably slightly inaccurate.

EXAMPLE 2. In an experiment with a Weston Differential Pulley Block, the effort E, in pounds, required to raise a load W, in pounds, was found to be as follows:

W	10	20	30	40	50	60	70	80	90	100
E	$3\frac{1}{4}$	$4\frac{7}{8}$	$6\frac{1}{4}$	$7\frac{1}{2}$	9	$10\frac{1}{2}$	$12\frac{1}{4}$	$13\frac{3}{4}$	15	$16\frac{1}{2}$

Find a relation of the form $E = aW + b$ that approximately agrees with these data. [GIBSON]

These values may be plotted in the usual manner on squared paper. They will be found to lie very nearly on a straight line. If E is plotted vertically, b is the intercept on the vertical axis, and a is the slope of the line; both can be measured directly in the figure.

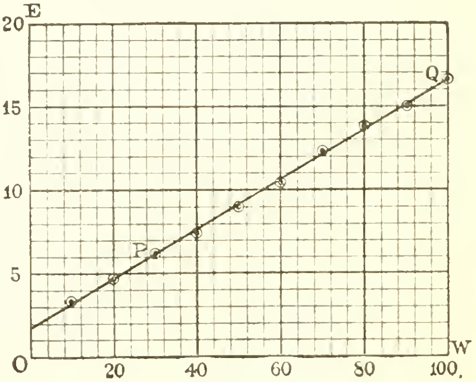


FIG. 118

To determine a and b more exactly, we may take various points that lie nearly on the line. Thus ($E = 6\frac{1}{4}$, $W = 30$) and ($E = 16\frac{1}{2}$, $W = 100$) lie nearly on a line that passes close to all the points. Substituting in the equation $E = aW + b$ we obtain

$$6\tfrac{1}{4} = 30a + b, \qquad 16\tfrac{1}{2} = 100a + b$$

whence $a = 0.146$, $b = 1.86$. Hence we may take

$$E = 0.146\,W + 1.86$$

approximately. Other pairs of values of E and W may be used in like manner to find values for a and b, and all the values of each quantity may be averaged.

Example 3. If θ denotes the melting point (Centigrade) of an alloy of lead and zinc containing x per cent of lead, it is found that

$x =$ % lead	40	50	60	70	80	90
$\theta =$ melting point	186°	205°	226°	250°	276°	304°

Find a relation of the form $\theta = a + bx + cx^2$ that approximately expresses these facts. [Saxelby]

Taking any three pairs of values, say (40, 186), (70, 250), (90, 304), and substituting in $\theta = a + bx + cx^2$ we find

$$186 = a + 40\,b + 1600\,c,$$
$$250 = a + 70\,b + 4900\,c,$$
$$304 = a + 90\,b + 8100\,c,$$

whence $a = 132$, $b = .92$, $c = .0011$, approximately; whence

$$\theta = 132 + .92\,x + .0011\,x^2.$$

Other sets of three pairs of values of x and y may be used in a similar manner to determine a, b, c; and the resulting values averaged, as above.

EXERCISES

1. In experiments on an iron rod, the amount of elongation l (in thousandths of an inch) and the stretching force p (in thousands of pounds) were found to be ($p = 10$, $l = 8$), ($p = 20$, $l = 15$), ($p = 40$, $l = 31$). Find a formula of the type $l = k \cdot p$ which approximately expresses these data. *Ans.* $k = .775$.

2. The values 1 in. = 2.5 cm. and 1 ft. = 30.5 cm. are frequently quoted, but they do not agree precisely. The number of centimeters, c, in i inches is surely given by a formula of the type $c = ki$. Find k approximately from the preceding data.

3. The readings of a standard gas-meter S and those of a meter T being tested on the same pipe-line were found to be ($S = 3000$, $T = 0$), ($S = 3510$, $T = 500$), ($S = 4022$, $T = 1000$). Find a formula of the type $T = aS + b$ which approximately represents these data.

4. An alloy of tin and lead containing x per cent of lead melts at the temperature θ (Fahrenheit) given by the values ($x = 25\%$, $\theta = 482°$), ($x = 50\%$, $\theta = 370°$), ($x = 75\%$, $\theta = 356°$). Determine a formula of the type $\theta = a + bx + cx^2$ which approximately represents these values.

5. The temperatures θ (Centigrade) at a depth d (feet) below the surface of the earth in a mine were found to be $d = 100$, $\theta = 15.7°$; $d = 200$, $\theta = 16.5$; $d = 300$, $\theta = 17.4$. Find a relation of the form $\theta = a + bd$ between θ and d.

6. Determine a line that passes reasonably near each of the three points (2, 4), (6, 7), (10, 9). Determine a quadratic expression $y = a + bx + cx^2$ that represents a parabola through the same three points.

7. Determine a parabola whose equation is of the form $y = a + bx + cx^2$ that passes through each of the points (0, 2.5), (1.5, 1.5), and (3.0, 2.8). Are the values of a, b, c changed materially if the point (2.0, 1.7) is substituted for the point (1.5, 1.5)?

8. If the curve $y = \sin x$ is drawn with one unit space on the x-axis representing 60°, the points (0, 0), $(\frac{1}{2}, \frac{1}{2})$, $(1\frac{1}{2}, 1)$ lie on the curve. Find a parabola of the form $y = a + bx + cx^2$ through these three points, and draw the two curves on the same sheet of paper to compare them.

285. Substitutions. It is particularly easy to test whether points that are given by an experiment really lie on a straight line; that is, whether the quantities measured satisfy an equation of the form $y = a + bx$. This is done by means of a transparent ruler or a stretched rubber band.

For this reason, if it is suspected that two quantities x and y satisfy an equation of the form

$$y = a + bx^2,$$

it is advantageous to substitute a new letter, say u, for x^2:

$$u = x^2, \quad y = a + bu$$

and then plot the values of y and u. If the new figure does agree reasonably well with some straight line, it is easy to find a and b, as in § 284.

Likewise, if it is suspected that two quantities x and y are connected by a relation of the form

$$y = a + b \cdot \frac{1}{x} \quad \text{or} \quad xy = ax + b,$$

it is advantageous to make the substitution $u = 1/x$.

Other substitutions of the same general nature are often useful.

In any case, the given values of x and y should be plotted first unchanged, in order to see what substitution might be useful.

286. Illustrative Example. If a body slides down an inclined plane, the distance s that it moves is connected with the time t after it starts by an equation of the form $s = kt^2$. Find a value of k that agrees reasonably with the following data :

s, in feet	2.6	10.1	23.0	40.8	63.7
t, in seconds	1	2	3	4	5

In this case, it is not necessary to plot the values of s and t themselves, because the nature of the equation, $s = kt^2$, is known from physics.

Hence we make the substitution $t^2 = u$, and write down the supplementary table :

s, in feet	2.6	10.1	23.0	40.8	63.7
u (or t^2)	1	4	9	16	25

These values will be found to give points very nearly on a straight line whose equation is of the form $s = ku$. To find k, we divide each value of s by the corresponding value of u ; this gives several values of k :

k	2.6	2.525	2.556	2.55	2.548

The *average* of these values of k is approximately 2.556 ; hence we may write $s = 2.556\,u$, or $s = 2.556\,t^2$.

EXERCISES

1. Find a formula of the type $u = kv^2$ that represents approximately the following values :

u	3.9	15.1	34.5	61.2	95.5	137.7	187.4
v	1	2	3	4	5	6	7

2. A body starts from rest and moves s feet in t seconds according to the following measured values:

s, in feet	3.1	13.0	30.6	50.1	79.5	116.4
t, in seconds	.5	1	1.5	2	2.5	3

Find approximately the relation between s and t.

3. The pressure p, measured in centimeters of mercury, and the volume v, measured in cubic centimeters, of a gas kept at constant temperature, were found to be:

v	145	155	165	178	191
p	117.2	109.4	102.4	95.0	88.6

Substitute u for $1/v$, compute the values of u, and determine a relation of the form $p = ku$; that is, $p = k/v$.

4. Determine a relation of the form $y = a + bx^2$ that approximately represents the values:

x	1	2	3	4	5	6	7
y	14.1	25.2	44.7	71.4	105.6	147.9	197.7

287. Logarithmic Plotting. In case the quantities y and x are connected by a relation of the form

$$y = kx^n,$$

it is advantageous to take logarithms (to the base 10) on both sides:

$$\log y = \log kx^n = \log k + n \log x,$$

and then substitute new letters for $\log x$ and $\log y$:

$$u = \log x, \qquad v = \log y.$$

For, if we do so, the equation becomes

$$v = l + nu,$$

where $l = \log k$.

If the values of x and y are given by an experiment, and if $u = \log x$ and $v = \log y$ are computed, the values of u and v should correspond to points that lie on a straight line, and the values of l and n can be found as in § 284. The value of k may be found from that of l, since $\log k = l$.

Example 1. The amount of water A, in cu. ft. that will flow per minute through 100 feet of pipe of diameter d, in inches, with an initial pressure of 50 lb. per sq. in., is as follows:

d	1	1.5	2	3	4	6
A	4.88	13.43	27.50	75.13	152.51	409.54

Find a relation between A and d.

Let $u = \log d$, $v = \log A$; then the values of u and v are

$u = \log d$ . . .	0.000	0.176	0.301	0.477	0.602	0.778
$v = \log A$ . . .	0.688	1.128	1.439	1.876	2.183	2.612

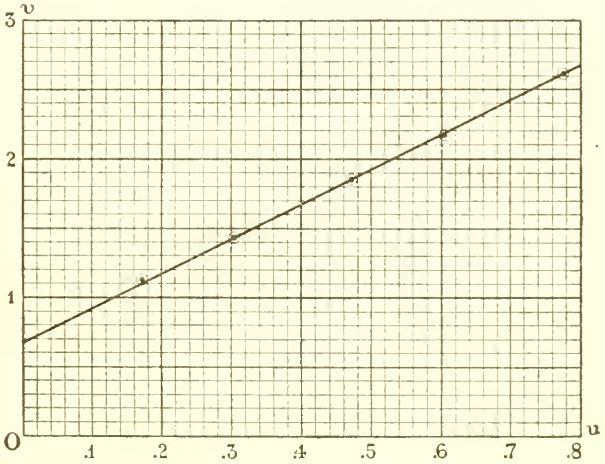


Fig. 119

These values give points in the (u, v) plane that are very nearly on a straight line; hence we may write, approximately,

$$v = a + bu,$$

where a and b can be determined directly by measurement in the figure, or as in § 284. If we take the first and last pairs of values of u and v, we find

$$.688 = a + 0.$$
$$2.612 = a + .778\,b.$$

Solving these equations, we find approximately, $a = .688$, $b = 2.473$, and we may write

$$v = .688 + 2.473\,u \quad \text{or} \quad \log A = .688 + 2.473 \log d.$$

Since $$.688 = \log 4.88,$$

the last equation may be written in the form

$$\log A = \log 4.88 + 2.473 \log d$$
$$= \log(4.88\,d^{2.473})$$

whence $$A = 4.88\,d^{2.473}.$$

Slightly different values of the constants may be found by using other pairs of values of u and v.

288. Logarithmic Paper. Paper called logarithmic paper may be bought that is ruled in lines whose distances, horizontally and vertically, from one point O (Fig. 120) are *proportional to the logarithms* of the numbers 1, 2, 3, etc.

Such paper may be used advantageously instead of actually looking up the logarithms in a table, as was done in § 287. For if the *given values* be plotted on this new paper, the resulting figure is identically the same as that obtained by plotting the *logarithms of the given values* on ordinary squared paper.

EXAMPLE. A strong rubber band stretched under a pull of p kg. shows an elongation of E cm. The following values were found in an experiment:

p	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0	6.0	7.0
E	0.1	0.3	0.6	0.9	1.3	1.7	2.2	2.7	3.3	3.9	5.3	6.9

[RIGGS]

If these values are plotted on logarithmic paper as in Fig. 120, it is evident that they lie reasonably near a straight line, such as that drawn.

By measurement in the figure, the slope of this line is found to be 1.6 approximately. Hence if $u = \log p$ and $v = \log E$ we have

$$v = l + 1.6\,u$$

where l is a constant not yet determined; whence

$$\log E = l + 1.6 \log p$$

or

$$E = kp^{1.6},$$

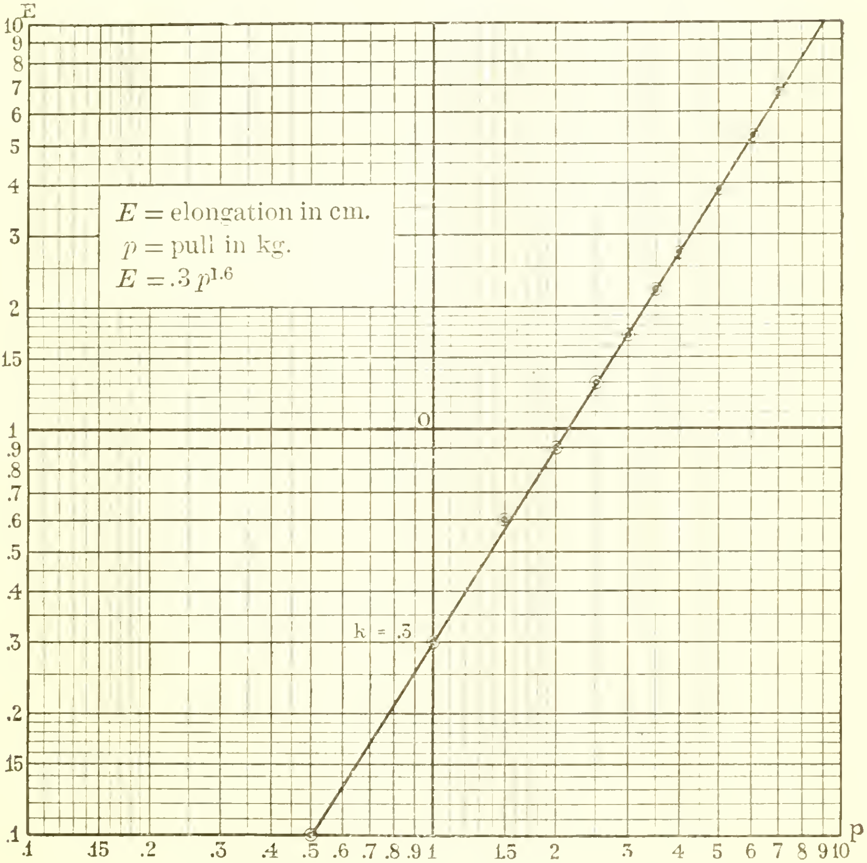


Fig. 120. — Elongation of a Rubber Band

where $l = \log k$. If $p = 1$, $E = k$; from the figure, if $p = 1$, $E = .3$; hence $k = .3$, and

$$E = .3\,p^{1.6}.$$

The use of logarithmic paper is however not at all essential; the same results may be obtained by the method of § 287.

EXERCISES

1. In testing a gas engine corresponding values of the pressure p, measured in pounds per square foot, and the volume v, in cubic feet, were obtained as follows: $v = 7.14$, $p = 54.6$; 7.73, 50.7; 8.59, 45.9. Find the relation between p and v (use logarithmic plotting).

Ans. $p = 387.6\ v^{-.938}$, or $pv^{.938} = 387.6$.

2. Expansion or contraction of a gas is said to be adiabatic when no heat escapes or enters. Determine the adiabatic relation between pressure p and volume v (Ex. 1) for air from the following observed values: $p = 20.54$, $v = 6.27$; 25.79, 5.34; 54.25, 3.15.

Ans. $pv^{1.41} = 273.5$.

3. The intercollegiate track records for foot-races are as follows, where d means the distance run, and t means the record time:

d	100 yd.	220 yd.	440 yd.	880 yd.	1 mi.	2 mi.
t	0:09$\frac{4}{5}$	0:21$\frac{1}{5}$	0:48	1:54$\frac{4}{5}$	4:15$\frac{2}{5}$	9:24$\frac{2}{5}$

Plot the logarithms of these values on squared paper (or plot the given values themselves on logarithmic paper). Find a relation of the form $t = kd^n$. What should be the record time for a race of 1320 yd.?

[See Kennelly, *Popular Science Monthly*, Nov. 1908.]

4. Solve the Example of § 288 by the method of § 287.

5. Each of the following sets of quantities was found by experiment. Find in each case an equation connecting the two quantities, by §§ 287-288.

(*a*)	v	1	2	3	4	5
	p	137.4	62.6	39.6	28.6	22.6

(*b*)	u	12.9	17.1	23.1	28.5	3.0
	v	63.0	27.0	13.8	8.5	6.9

(*c*)	θ	82°	212°	390°	570°	750°	1100°
	c	2.09	2.69	2.90	2.98	3.09	3.28

SOLID ANALYTIC GEOMETRY

CHAPTER XIII

COORDINATES

289. Location of a Point. The position of a point in three-dimensional space can be assigned without ambiguity by giving its distances from three mutually rectangular planes, provided these distances are taken with proper signs according as the point lies on one or the other side of each plane.

The three planes, each perpendicular to the other two, are called the ***coordinate planes;*** their common point O (Fig. 121) is called the ***origin.*** The three mutually rectangular lines Ox, Oy, Oz in which the planes intersect are called the ***axes*** of coordinates; on each of them a positive sense is selected arbitrarily, by affixing the letter x, y, z, respectively.

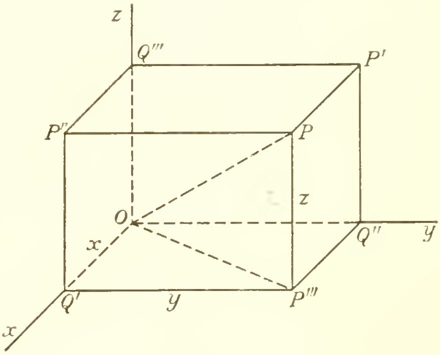


FIG. 121

The three coordinate planes, Oyz, Ozx, Oxy, divide the whole of space into eight compartments called ***octants.*** The first octant in which all three coordinates are positive is also called the ***coordinate trihedral.***

If P', P'', P''' are the projections of any point P on the coordinate planes Oyz, Ozx, Oxy, respectively, then $P'P = x$, $P''P = y$, $P'''P = z$ are the ***rectangular cartesian coordinates*** of

P. If the planes through P parallel to Oyz, Ozx, Oxy intersect the axes Ox, Oy, Oz in Q', Q'', Q''', the point P is found from its coordinates x, y, z by passing along the axis Ox through the distance $OQ' = x$, parallel to Oy through the distance $Q'P''' = y$, and parallel to Oz through the distance $P'''P = z$, each of these distances being taken with the proper sense.

Every point in space has three definite real numbers as coordinates; conversely, *to every set of three real numbers corresponds one and only one point.*

Locate the points: $(2, 3, 4)$, $(-3, 2, 0)$, $(5, 0, -3)$, $(0, 0, 4)$, $(0, -6, 0)$, $(-5, -8, -2)$.

290. Distance of a Point from the Origin. For the distance $OP = r$ (Fig. 121) of the point $P(x, y, z)$ from the origin O we have, since OP is the diagonal of a rectangular parallelepiped with edges $OQ' = x$, $OQ'' = y$, $OQ''' = z$:

$$r = \sqrt{x^2 + y^2 + z^2}.$$

291. Distance between two Points. The distance between the two points P_1 (x_1, y_1, z_1) and P_2 (x_2, y_2, z_2) can be found if the coordinates of the two points are given. For (Fig. 122), the planes through P_1 and those through P_2 parallel to the coordinate planes bound a rectangular parallelepiped with $P_1P_2 = d$ as diagonal; and as its edges are

$$P_1Q = x_2 - x_1, \quad P_1R = y_2 - y_1, \quad P_1S = z_2 - z_1,$$

we find

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

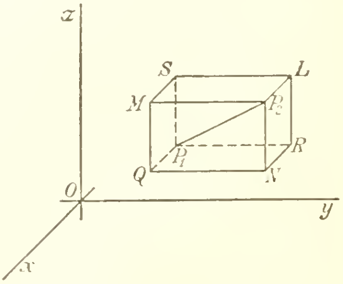


FIG. 122

292. Oblique Axes. The position of a point P in space can also be determined with respect to three axes *not* at right angles. The coordinates of P are the segments cut off on the axes by planes through P

parallel to the coordinate planes. In what follows, the axes are always assumed to be at right angles unless the contrary is definitely stated.

EXERCISES

1. What are the coordinates of the origin? What can you say of the coordinates of a point on the axis Ox? on the axis Oy? on the axis Oz?

2. What can you say of the coordinates of a point that lies in the plane Oxy? in the plane Oyz? in the plane Ozx?

3. Where is a point situated when $x = 0$? when $z = 0$? when $x = y = 0$? when $y = z$? when $x = 2$? when $z = -3$? when $x = 1$, $y = 2$?

4. A rectangular parallelepiped lies in the first octant with three of its faces in the coordinate planes, its edges are of length a, b, c respectively; what are the coordinates of the vertices?

5. Show that the points $(4, 3, 5)$, $(2, -1, 3)$, $(0, 1, 7)$ are the vertices of an equilateral triangle.

6. Show that the points $(-1, 1, 3)$, $(-2, -1, 4)$, $(0, 0, 5)$ lie on a sphere whose center is $(2, -3, 1)$. What is the radius of this sphere?

7. Show that the points $(6, 2, -5)$, $(2, -4, 7)$, $(4, -1, 1)$ lie on a straight line.

8. Show that the triangle whose vertices are (a, b, c), (b, c, a), (c, a, b) is equilateral.

9. What are the coordinates of the projections of the point $(6, 3, -8)$ on the axes of coordinates? What are the distances of this point from the coordinate axes?

10. What is the length of the segment of a line whose projections on the coordinate axes are 5, 3, and 2?

11. What are the coordinates of the points which are symmetric to the point (a, b, c) with respect to the coordinate planes? with respect to the axes? with respect to the origin?

12. Show that the sum of the squares of the four diagonals of a rectangular parallelepiped is equal to the sum of the squares of its edges.

293. Projection. The ***projection*** of a point on a plane or line is the foot of the perpendicular let fall from the point on the plane or line. The projection of a rectilinear segment AB on a plane or line is the intercept $A'B'$ between the feet of the perpendiculars AA', BB' let fall from A, B on the plane or line. If α is one of the two angles made by the segment with the plane or line we have

$$A'B' = AB \cos \alpha.$$

In analytic geometry we have generally to project a ***vector***, *i.e.* a segment with a definite sense, on an *axis*, *i.e.* on a line with a definite sense (compare § 19). The angle α is then understood to be the angle between the positive senses of vector and axis (both being drawn from a common origin). The above formula then gives the projection with its proper sign.

Thus, the segment OP (Fig. 121) from the origin to any point $P(x, y, z)$ can be regarded as a vector OP. Its projections on the axes of coordinates are the coordinates x, y, z of P. These projections are also called the *rectangular components* of the vector OP, and OP is called the *resultant* of the components OQ', OQ'', OQ''', or also of OQ', $Q'P'''$, $P'''P$.

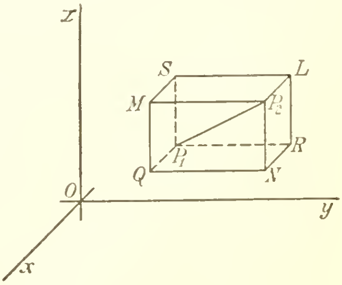


FIG. 123

Similarly, in Fig. 123, if P_1P_2 be regarded as a vector, the projections of this vector P_1P_2 on the axes of coordinates are the coordinate differences $x_2 - x_1$, $y_2 - y_1$, $z_2 - z_1$. See § 298.

294. Resultant. The proposition of § 19 that *the sum of the projections of the sides of an open polygon on any axis is* *equal to the projection of the closing side on the same axis* and that of § 20 that *the projection of the resultant is equal to the sum of the projections of its components* are readily seen to hold in three dimensions as well as in the plane. Analytically these propositions follow by considering that whatever the points $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$, $\cdots$ $P_n(x_n, y_n, z_n)$ in space, the sum of the projections of the vectors P_1P_2, P_2P_3, $\cdots$ $P_{n-1}P_n$ on the axis Ox is:

$$(x_2 - x_1) + (x_3 - x_2) + \cdots + (x_n - x_{n-1}) = x_n - x_1,$$

where the right-hand member is the projection of the closing side or resultant P_1P_n on Ox. Any line can of course be taken as axis Ox.

295. Division Ratio. *Two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ being given by their coordinates, the coordinates x, y, z of any point P of the line P_1P_2 can be found if the division ratio $P_1P/P_1P_2 = k$ is known in which the point P divides the segment P_1P_2* (Fig 124).

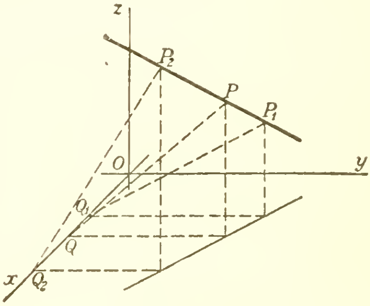


FIG. 124

Let Q_1, Q, Q_2 be the projections of P_1, P, P_2 on the axis Ox; as Q divides Q_1Q_2 in the same ratio k in which P divides P_1P_2, we have as in § 3:

$$x = x_1 + k(x_2 - x_1).$$

Similarly we find by projecting on Oy, Oz:

$$y = y_1 + k(y_2 - y_1), \quad z = z_1 + k(z_2 - z_1).$$

If k is positive, P lies on the same side of P_1 as does P_2; if k is negative, P lies on the opposite side of P_1 (§ 3).

296. Direction Cosines. Instead of using the cartesian coordinates x, y, z to locate a point P (Fig. 125) we can also use its ***radius vector*** $r = OP$, *i.e.* the length of the vector drawn from the origin to the point, and its ***direction cosines***, *i.e.* the cosines of the angles α, β, γ, made by the vector OP with the axes Ox, Oy, Oz. We have evidently

$$\boldsymbol{x = r\cos\alpha,\quad y = r\cos\beta,\quad z = r\cos\gamma.}$$

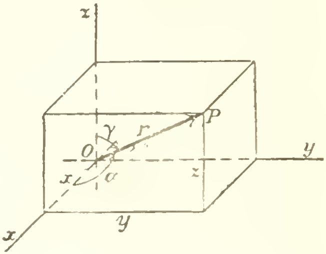


FIG. 125

As a line has two opposite senses we can take as *direction cosines of any line* parallel to OP either $\cos\alpha$, $\cos\beta$, $\cos\gamma$, or $-\cos\alpha$, $-\cos\beta$, $-\cos\gamma$.

The direction cosines $\cos\alpha$, $\cos\beta$, $\cos\gamma$ of a vector OP are often denoted briefly by the letters l, m, n, respectively, so that the coordinates of P are

$$x = lr,\quad y = mr,\quad z = nr.$$

The direction cosines of any parallel line are then l, m, n or $-l$, $-m$, $-n$.

297. Pythagorean Relation. *The sum of the squares of the direction cosines of any line is equal to one.*

For, the equations of § 347 give upon squaring and adding since $x^2 + y^2 + z^2 = r^2$:

$$\boldsymbol{\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1,}$$

or

$$l^2 + m^2 + n^2 = 1;$$

and this still holds when l, m, n are replaced by $-l$, $-m$, $-n$. Since this result is derived directly from the Pythagorean Theorem of geometry, it may be called the *Pythagorean Relation* between the direction cosines. Notice that l, m, n can be regarded as the coordinates of the extremity of a vector of unit length drawn from the origin parallel to the line.

EXERCISES

1. Find the length of the radius vector and its direction cosines for each of the following points: $(5, -3, 2)$; $(-3, -2, 1)$; $(-4, 0, 8)$.

2. The direction cosines of a line are proportional to 1, 2, 3; find their values.

3. A straight line makes an angle of $30°$ with the axis Ox and an angle of $60°$ with the axis Oy; what is the third direction angle?

4. What is the direction of a line when $l = 0$? when $l = m = 0$?

5. What are the direction cosines of that line whose direction angles are equal?

6. What are the direction cosines of the line bisecting the angle between two intersecting lines whose direction cosines are l, m, n and l', m', n', respectively?

7. Find the direction cosines of the line which bisects the angle between the radii vectores of the points $(3, -4, 2)$ and $(-1, 2, 3)$.

8. Three vertices of a parallelogram are $(4, 3, -2)$, $(7, -1, 4)$, $(-2, 1, -4)$; find the coordinates of the fourth vertex (three solutions).

9. In what ratio is the line drawn from the point $(2, -5, 8)$ to the point $(4, 6, -2)$ divided by the plane Ozx? by the plane Oxy? At what points does this line pierce these coordinate planes?

10. In what ratio is the line drawn from the point $(0, 5, 0)$ to the point $(8, 0, 0)$ divided by the line in the plane Oxy which bisects the angle between the axes?

11. Find the coordinates of the midpoint of the line joining the points $(4, -3, 8)$ and $(6, 5, -9)$. Find the points which trisect the same segment.

12. If we add to the segment joining the points $(4, 1, 2)$ and $(-2, 5, 7)$ a segment of twice its length in each direction, what are the coordinates of the end points?

13. Find the coordinates of the intersection of the medians of the triangle whose vertices are $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$, $P_3(x_3, y_3, z_3)$.

14. Show that the lines joining the midpoints of the opposite edges of a tetrahedron intersect and are bisected by their common point.

15. Show that the projection of the radius vector of the point $P(x, y, z)$ on a line whose direction cosines are l', m', n' is $l'x + m'y + n'z$.

298. Projections. Components of a Vector. If two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ are given by their coordinates, the *projections of the vector*, P_1P_2 on the axes, or what amounts to the same, on parallels to the axes drawn through P_1 (Fig. 126), are evidently (§ 293):

$$P_1Q = x_2 - x_1, \quad P_1R = y_2 - y_1,$$
$$P_1S = z_2 - z_1.$$

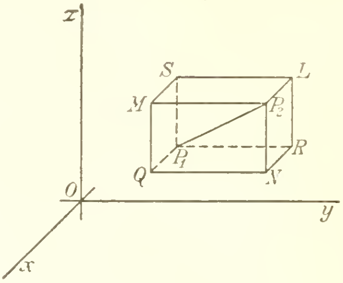


FIG. 126

These projections, or also the vectors P_1Q, QN, NP_2, are called the *rectangular components* of the vector P_1P_2, or its *components along the axes.*

If d is the length of the segment P_1P_2, its direction cosines l, m, n are since P_2Q is perpendicular to P_1Q, P_2R to P_1R, P_2S to P_1S:

$$l = \frac{x_2 - x_1}{d}, \quad m = \frac{y_2 - y_1}{d}, \quad n = \frac{z_2 - z_1}{d}.$$

These relations can also be written in the form:

$$\frac{x_2 - x_1}{l} = \frac{y_2 - y_1}{m} = \frac{z_2 - z_1}{n} = d.$$

299. Angle between two Lines. *If the directions of two lines are given by their direction cosines l_1, m_1, n_1 and l_2, m_2, n_2, the angle ψ between the two lines is given by the formula*

$$\cos\psi = l_1l_2 + m_1m_2 + n_1n_2.$$

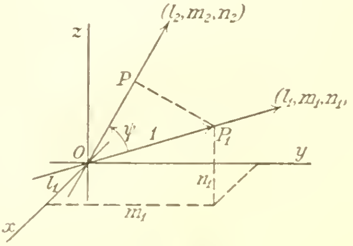


FIG. 127

For, drawing through the origin two lines of direction cosines l_1, m_1, n_1 and l_2, m_2, n_2 and taking on the former a vector OP_1 of unit length, the projection OP of OP_1 on the other line is equal to the cosine of the required angle ψ. On the other hand, OP_1 has l_1, m_1, n_1 as components along the axes; hence, by § 294:

$$\cos\psi = l_1l_2 + m_1m_2 + n_1n_2.$$

Two intersecting lines (or any two parallels to them) make two angles, say ψ and $\pi - \psi$. But if the direction cosines of each line are given, a definite sense has been assigned to each line, and the angle between the lines is understood to be the angle between these senses.

300. Conditions for Parallelism and for Perpendicularity. If, in particular, the lines are parallel, we have either $l_1 = l_2$, $m_1 = m_2$, $n_1 = n_2$, or $l_1 = -l_2$, $m_1 = -m_2$, $n_1 = -n_2$; hence in either case

$$\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}.$$

This then is the ***condition of parallelism*** of two lines whose direction cosines are l_1, m_1, n_1 and l_2, m_2, n_2.

If the lines are perpendicular, *i.e.* if $\psi = \frac{1}{2}\pi$, we have $\cos\psi = 0$; hence the ***condition of perpendicularity*** of two lines whose direction cosines are l_1, m_1, n_1 and l_2, m_2, n_2 is

$$l_1l_2 + m_1m_2 + n_1n_2 = 0.$$

301. The formula of § 299 gives

$$\sin^2\psi = 1 - \cos^2\psi = 1 - (l_1l_2 + m_1m_2 + n_1n_2)^2.$$

As (§ 297) $(l_1^2 + m_1^2 + n_1^2)(l_2^2 + m_2^2 + n_2^2) = 1$, we can write this expression in the form

$$\sin^2\psi = \begin{vmatrix} l_1^2 + m_1^2 + n_1^2 & l_1l_2 + m_1m_2 + n_1n_2 \\ l_1l_2 + m_1m_2 + n_1n_2 & l_2^2 + m_2^2 + n_2^2 \end{vmatrix},$$

which, by Ex. 3, p. 45, can also be expressed as follows:

$$\sin^2\psi = \begin{vmatrix} m_1 & n_1 \\ m_2 & n_2 \end{vmatrix}^2 + \begin{vmatrix} n_1 & l_1 \\ n_2 & l_2 \end{vmatrix} + \begin{vmatrix} l_1 & m_1 \\ l_2 & m_2 \end{vmatrix}^2.$$

The *direction (l, m, n) perpendicular to two given different directions* (l_1, m_1, n_1) and (l_2, m_2, n_2) is found by solving the equations (§ 300)

$$l_1l + m_1m + n_1n = 0,$$
$$l_2l + m_2m + n_2n = 0,$$

whence
$$\frac{l}{\begin{vmatrix} m_1 & n_1 \\ m_2 & n_2 \end{vmatrix}} = \frac{m}{\begin{vmatrix} n_1 & l_1 \\ n_2 & l_2 \end{vmatrix}} = \frac{n}{\begin{vmatrix} l_1 & m_1 \\ l_2 & m_2 \end{vmatrix}}.$$

If we denote by k the common value of these ratios, we have

$$l = \begin{vmatrix} m_1 & n_1 \\ m_2 & n_2 \end{vmatrix} k, \quad m = \begin{vmatrix} n_1 & l_1 \\ n_2 & l_2 \end{vmatrix} k, \quad n = \begin{vmatrix} l_1 & m_1 \\ l_2 & m_2 \end{vmatrix} k;$$

substituting these values in the relation (§ 297) $l^2 + m^2 + n^2 = 1$ and, observing the preceding value of $\sin\psi$, we find:

$$l = \pm \frac{\begin{vmatrix} m_1 & n_1 \\ m_2 & n_2 \end{vmatrix}}{\sin\psi}, \quad m = \pm \frac{\begin{vmatrix} n_1 & l_1 \\ n_2 & l_2 \end{vmatrix}}{\sin\psi}, \quad n = \pm \frac{\begin{vmatrix} l_1 & m_1 \\ l_2 & m_2 \end{vmatrix}}{\sin\psi},$$

where ψ is the angle between the given directions.

302. Three directions (l_1, m_1, n_1), (l_2, m_2, n_2), (l_3, m_3, n_3) are ***complanar***, *i.e.* parallel to the same plane, if there exists a direction (l, m, n) perpendicular to all three. This will be the case if the equations

$$l_1 l + m_1 m + n_1 n = 0,$$
$$l_2 l + m_2 m + n_2 n = 0,$$
$$l_3 l + m_3 m + n_3 n = 0$$

have solutions not all zero; hence the *condition of complanarity*

$$\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} = 0.$$

EXERCISES

1. Find the length and direction cosines of the vector drawn from the point $(5, -2, 1)$ to the point $(4, 8, -6)$; from the point (a, b, c) to the point $(-a, -b, -c)$; from $(-a, -b, -c)$ to (a, b, c).

2. Show that when two lines with direction cosines l, m, n and l', m', n', respectively, are parallel, $ll' + mm' + nn' = \pm 1$.

3. Show that when two lines with direction cosines proportional to a, b, c, and a', b', c', are perpendicular $aa' + bb' + cc' = 0$; and when the lines are parallel $a/a' = b/b' = c/c'$.

4. Show that the points $(5, 2, -3)$, $(6, 1, 4)$, $(-2, -3, 6)$, $(-1, -4, 13)$ are the vertices of a parallelogram.

5. Show by direction cosines that the points $(6, -3, 5)$, $(8, 2, 2)$, $(4, -8, 8)$ lie in a line.

6. Find the angle between the vectors from $(5, 8, -2)$ to $(-2, 6, -1)$ and from $(8, 3, 5)$ to $(1, 1, -6)$.

7. Find the angles of the triangle whose vertices are $(5, 2, 1)$, $(0, 3, -1)$, $(2, -1, 7)$.

8. Find the direction cosines of a line which is perpendicular to two lines whose direction cosines are proportional to $2, -3, 4$, and $5, 2, -1$, respectively.

9. Derive the formula of § 299 by taking on each line a vector of unit length, OP_1 and OP_2, and expressing the distance P_1P_2 first by the cosine law of trigonometry, then by § 291, and equating these expressions.

10. Find the rectangular components of a force of 12 lb. acting along a line inclined at 60° to Ox and at 45° to Oy.

11. Find the resultant of the forces OP_1, OP_2, OP_3, OP_4 if the coordinates of P_1, P_2, P_3, P_4, with O as origin, are $(3, -1, 2)$, $(2, 2, -1)$, $(-1, 2, 1)$, $(-2, 3, -4)$.

12. If any number of vectors, applied at the origin, are given by the coordinates x, y, z of their extremities, the length of the resultant R is $\sqrt{(\Sigma x)^2 + (\Sigma y)^2 + (\Sigma z)^2}$ (see Ex. 9, p. 21), and its direction cosines are $\Sigma x/R$, $\Sigma y/R$, $\Sigma z/R$.

13. A particle at one vertex of a cube is acted upon by seven forces represented by the vectors from the particle to the other seven vertices; find the magnitude (length) and direction of the resultant.

14. If four forces acting on a particle are parallel and proportional to the sides of a quadrilateral, the forces are in equilibrium, *i.e.* their resultant is zero. Similarly for any closed polygon.

303. Translation of Coordinate Trihedral. Let x, y, z be the coordinates of any point P with respect to the trihedral formed by the axes Ox, Oy, Oz (Fig. 128). If parallel axes O_1x_1, O_1y_1, O_1z_1 be drawn through any point $O_1(a, b, c)$, and if x_1, y_1, z_1 are the coordinates of P with respect to the new trihedral $O_1x_1y_1z_1$, then the relations between the old coordinates x, y, z, and the new coordinates x_1, y_1, z_1 of one and the same point P are evidently

$$x = a + x_1, \quad y = b + y_1, \quad z = c + z_1.$$

The coordinate trihedral has thus been given a *translation*, represented by the vector OO_1. This operation is also called a *transformation to parallel axes* through O_1.

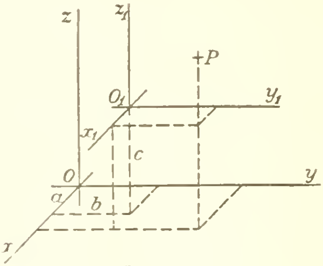


FIG. 128

304. Area of a Triangle. Any two vectors OP_1, OP_2 drawn from the origin determine a triangle OP_1P_2, whose area A can easily be expressed if the lengths r_1, r_2 and direction cosines of the vectors are given. For, denoting the angle P_1OP_2 by ψ we have for the area A:

$$A = \tfrac{1}{2} r_1r_2 \sin\psi,$$

where $\sin\psi$ can be expressed in terms of the direction cosines by § 301.

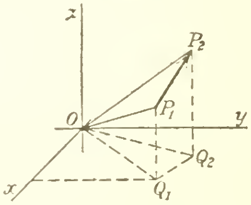


FIG. 129

305. Moment of a Force. Such areas are used in mechanics to represent the *moments* of forces. The moment of a force about a point O is defined as the product of the force into the perpendicular distance of O from the line of action of the force. Thus, if the vector P_1P_2 (Fig. 130) represent a force (in magnitude, direction, and sense) the moment of this force about the origin O is equal to twice the area of the triangle OP_1P_2, *i.e.* to the area of the parallelogram $OP_1P_2P_3$, where OP_3 is a vector equal to the vector P_1P_2.

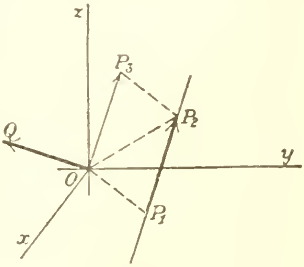


FIG. 130

It is often more convenient to represent this moment not by such an area, but by a *vector* OQ, drawn from O at right angles to the triangle, and of a length equal to the number that represents the moment. If the body on which the force acts could turn freely about this perpendicular the moment would represent the turning effect of the force P_1P_2.

The *sense* of this vector that represents the moment is taken so as to make the vector point toward that side of the plane of the triangle from which the force P_1P_2 is seen to turn counterclockwise.

306. If we square the expression found in § 304 for the area of the triangle OP_1P_2 and substitute for $\sin^2\psi$ its value from § 301, we find:

$$A^2 = \tfrac{1}{4} r_1^2 r_2^2 \left(\begin{vmatrix} m_1 & n_1 \\ m_2 & n_2 \end{vmatrix}^2 + \begin{vmatrix} n_1 & l_1 \\ n_2 & l_2 \end{vmatrix}^2 + \begin{vmatrix} l_1 & m_1 \\ l_2 & m_2 \end{vmatrix}^2 \right).$$

Hence A^2 is the sum of the squares of the three quantities

$$A_x = \tfrac{1}{2} r_1 r_2 \begin{vmatrix} m_1 & n_1 \\ m_2 & n_2 \end{vmatrix}, \quad A_y = \tfrac{1}{2} r_1 r_2 \begin{vmatrix} n_1 & l_1 \\ n_2 & l_2 \end{vmatrix}, \quad A_z = \tfrac{1}{2} r_1 r_2 \begin{vmatrix} l_1 & m_1 \\ l_2 & m_2 \end{vmatrix},$$

which have a simple geometrical and mechanical interpretation. For, as the coordinates of P_1, P_2 are

$$x_1 = l_1 r_1, \quad y_1 = m_1 r_1, \quad z_1 = n_1 r_1,$$
$$x_2 = l_2 r_2, \quad y_2 = m_2 r_2, \quad z_2 = n_2 r_2,$$

we have, *e.g.*,

$$A_z = \tfrac{1}{2} \begin{vmatrix} l_1 r_1 & m_1 r_1 \\ l_2 r_2 & m_2 r_2 \end{vmatrix} = \tfrac{1}{2} \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix};$$

and as x_1, y_1 and x_2, y_2 are the coordinates of the projections Q_1, Q_2 of P_1, P_2 on the plane Oxy, A_z represents (§ 12) the area of the triangle OQ_1Q_2, *i.e. the projection on the plane Oxy of the area OP_1P_2*. Similarly, A_x and A_y are the projections of the area OP_1P_2 on the planes Oyz and Ozx, respectively. As any three mutually rectangular planes can be taken as coordinate trihedrals, our formula $A^2 = A_x^2 + A_y^2 + A_z^2$ means that *the square of the area of any triangle is equal to the sum of the squares of its projections on any three mutually rectangular planes.*

In mechanics, $2A_z$ is the moment of the projection Q_1Q_2 of the force P_1P_2 about O, or what is by definition the same thing, the *moment of P_1P_2 about the axis Oz.* Similarly, for $2A_x$, $2A_y$. The proposition means, therefore, that the moments of P_1P_2 about the axes Ox, Oy, Oz laid off as vectors along these axes can be regarded as the rectangular components of the moment of P_1P_2 about the point O; in other words, $2A_x$, $2A_y$, $2A_z$ are the components along Ox, Oy, Oz of that vector $2A$ (§ 305) which represents the moment of P_1P_2 about O.

307. Polar Coordinates. The position of any point P (Fig. 131) can also be assigned by its *radius vector* $OP = r$, *i.e.* the distance of P from a fixed origin or *pole* O, and two angles: the *colatitude* θ, *i.e.* the angle NOP made by OP with a fixed axis ON, the *polar axis*, and the *longitude* ϕ, *i.e.* the angle AOP' made by the plane of θ with a fixed plane NOA through the polar axis, the *initial meridian plane*.

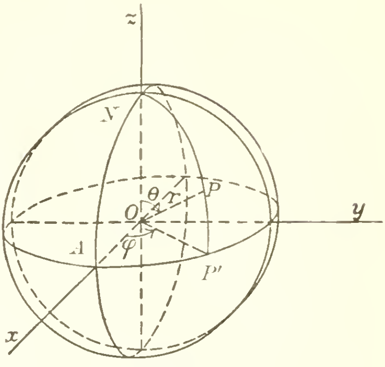


FIG. 131

A given radius vector r confines the point P to the sphere of radius r about the pole O. The angles θ and ϕ serve to determine the position of P on this sphere. This is done as on the earth's surface except that instead of the *latitude*, which is the angle made by the radius vector with the plane of the equator AP', we use the colatitude or polar distance $\theta = NOP$.

The quantities r, θ, and ϕ are the *polar* or *spherical coordinates* of P. After assuming a point O as pole, a line ON through O, with a definite sense, as polar axis, and a (half-) plane through this axis as initial meridian plane, every point P has a definite radius vector r (varying from zero to infinity), colatitude θ (varying from 0 to π), and a definite longitude ϕ (varying from 0 to 2π). The counterclockwise sense of rotation about the polar axis is taken as the positive sense of ϕ.

308. Transformation from Cartesian to Polar Coordinates. The relations between the cartesian coordinates x, y, z and the polar coordinates r, θ, ϕ of any point P appear directly from Fig. 132. If the axis Oz coincides with the polar axis, the plane Oxy with the *equatorial plane*, *i.e.* the plane through the pole at right angles to the polar axis, while the plane Ozx is taken as initial meridian plane, the projections of $OP = r$ on the axis Oz and on the equatorial plane are

$$OR = r\cos\theta,\quad OQ = r\sin\theta.$$

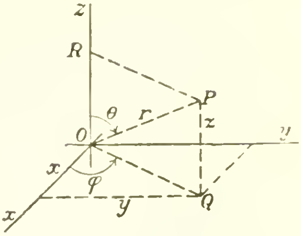


FIG. 132

Projecting OQ on the axes Ox, Oy, we find

$$x = r\sin\theta\cos\phi,\quad y = r\sin\theta\sin\phi,\quad z = r\cos\theta.$$

Also $$r = \sqrt{x^2+y^2+z^2},\ \cos\theta = \frac{z}{\sqrt{x^2+y^2+z^2}},\ \tan\phi = \frac{y}{x}.$$

EXERCISES

1. Find the area of the triangle whose vertices are $(a, 0, 0)$, $(0, b, 0)$, $(0, 0, c)$.

2. Find the area of the triangle whose vertices are the origin and the points $(3, 4, 7)$, $(-1, 2, 4)$.

3. Find the area of the triangle whose vertices are $(4, -3, 2)$, $(6, 4, 4)$, $(-5, -2, 8)$.

4. The cartesian coordinates of a point are 1, $\sqrt{3}$, $2\sqrt{3}$; what are its polar coordinates?

5. If $r = 5$, $\theta = \frac{1}{3}\pi$, $\phi = \frac{1}{6}\pi$, what are the cartesian coordinates?

6. The earth being taken as a sphere of radius 3962 miles, what are the polar and cartesian coordinates of a point on the surface in lat. 42° 17′ N. and long. 83° 44′ W. of Greenwich, the north polar axis being the axis Oz and the initial meridian passing through Greenwich? What is the distance of this point from the earth's axis?

7. Find the area of the triangle whose vertices are $(0, 0, 0)$, (r_1, θ_1, ϕ_1), (r_2, θ_2, ϕ_2).

8. Express the distance between any two points in polar coordinates.

9. Find the area of any triangle when the cartesian coordinates of the vertices are given.

10. Find the rectangular components of the moment about the origin of the vector drawn from $(1, -2, 3)$ to $(3, 1, -1)$.

CHAPTER XIV

THE PLANE AND THE STRAIGHT LINE

PART I. THE PLANE

309. Locus of One Equation. In plane analytic geometry any equation between the coordinates x, y or r, ϕ of a point in general represents a plane curve. In particular, an equation of the first degree in x and y represents a straight line (§ 30); an equation of the second degree in x and y in general represents a conic section (§ 245).

In solid analytic geometry any equation between the coordinates x, y, z or r, θ, ϕ of a point in general represents a ***surface***. Thus, if any equation in x, y, z,

$$F(x, y, z) = 0,$$

be imagined solved for z so as to take the form

$$z = f(x, y),$$

we can find from this equation to every point (x, y) in the plane Oxy one or more ordinates z (which may of course be real or imaginary), and the ***locus*** formed by the extremities of the real ordinates will in general form a surface. It may however happen *in particular cases* that the locus of the equation $F(x, y, z) = 0$, *i.e.* the totality of all those points whose coordinates x, y, z when substituted in the equation satisfy it, consists only of isolated points, or forms a curve, or that there are no real points satisfying the equation.

Similar considerations apply to an equation in polar coordinates

$$F(r, \theta, \phi) = 0.$$

310. Locus of Two Simultaneous Equations. Two simultaneous equations in x, y, z (or in the polar coordinates r, θ, ϕ) will in general represent a *curve* in space, namely, the intersection of the two surfaces represented by the two equations separately.

Thus, in the present chapter, we shall see that an equation of the first degree in x, y, z represents a plane and that therefore two such equations represent a straight line, the intersection of the two planes. In chapters XV and XVI we shall discuss loci represented by equations of the second degree, which are called *quadric surfaces*.

311. Equation of a Plane. *Every equation of the first degree in x, y, z represents a plane.* The plane is defined as a surface such that the line joining any two of its points lies completely in the surface. We have therefore to show that if the general equation of the first degree

$$(1) \qquad Ax + By + Cz + D = 0$$

is satisfied by the coordinates of any two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$, *i.e.* if

$$(2) \qquad \begin{cases} Ax_1 + By_1 + Cz_1 + D = 0, \\ Ax_2 + By_2 + Cz_2 + D = 0, \end{cases}$$

then (1) is satisfied by the coordinates of *every* point $P(x, y, z)$ of the line P_1P_2.

Now, by § 295, the coordinates of every point of the line P_1P_2 can be expressed in the form

$$x = x_1 + k(x_2 - x_1), \ y = y_1 + k(y_2 - y_1), \ z = z_1 + k(z_2 - z_1),$$

where k is the ratio in which P divides P_1P_2, *i.e.*

$$k = P_1P/P_1P_2.$$

We have therefore to show that

$$A[x_1 + k(x_2 - x_1)] + B[y_1 + k(y_2 - y_1)] + C[z_1 + k(z_2 - z_1)] + D = 0.$$

whatever the value of k. Adding and subtracting kD, we can write this equation in the form

$$(1-k)(Ax_1+By_1+Cz_1+D)+k(Ax_2+By_2+Cz_2+D)=0;$$

and this is evidently true for any k, owing to the conditions (2).

312. Essential Constants. The equation (1) will still represent the same plane when multiplied by any constant different from zero. Since A, B, C cannot all three be zero, we can divide (1) by one of these constants; it will then contain not more than three arbitrary constants. We say therefore that the general equation of a plane contains *three essential constants*. This corresponds to the geometrical fact that a plane can, in a variety of ways, be determined by three conditions, such as the conditions of passing through three points, etc.

313. Special Cases. If, in equation (1), $D=0$, the plane evidently passes through the origin.

If, in equation (1), $C=0$, so that the equation is of the form

$$Ax+By+D=0,$$

this equation represents the plane perpendicular to the plane Oxy and passing through the line whose equation in the plane Oxy is $Ax+By+D=0$. For, the equation $Ax+By+D=0$ is satisfied by the coordinates of all points (x, y, z) whose x and y are connected by the relation $Ax+By+D=0$ and whose z is arbitrary, but it is not satisfied by the coordinates of any other points. Similarly, if $B=0$ in (1), the plane is perpendicular to Ozx; if $A=0$, the plane is perpendicular to Oyz.

If $B=0$ and $C=0$ in (1), the equation obviously represents a plane perpendicular to the axis Ox; and similarly when C and A, or A and B are zero.

Notice that the line of intersection of (1) with the plane Oxy, for instance, is represented by the simultaneous equations

$$Ax + By + Cz + D = 0,\ z = 0.$$

314. Intercept Form. If $D \neq 0$ the equation (1) can be divided by D: it then assumes the form

$$\frac{A}{D}x + \frac{B}{D}y + \frac{C}{D}z + 1 = 0.$$

If A, B, C are all different from zero, this equation can be written

$$\frac{x}{-D/A} + \frac{y}{-D/B} + \frac{z}{-D/C} = 1,$$

or, putting $-D/A = a$, $-D/B = b$, $-D/C = c$:

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1. \tag{3}$$

In this equation, called the ***intercept form*** of the equation of a plane, the constants a, b, c are the intercepts made by the plane on the axes Ox, Oy, Oz respectively. For, putting, for instance, $y = 0$ and $z = 0$, we find $x = a$; etc.

315. Plane through Three Points. If the plane

$$Ax + By + Cz + D = 0$$

is to pass through the three points $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$, $P_3(x_3, y_3, z_3)$, the three conditions

$$Ax_1 + By_1 + Cz_1 + D = 0,$$
$$Ax_2 + By_2 + Cz_2 + D = 0,$$
$$Ax_3 + By_3 + Cz_3 + D = 0$$

must be satisfied. Eliminating A, B, C, D between the four linear homogeneous equations (compare § 75) we find the equation of the plane passing through the three points in the form

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0.$$

EXERCISES

1. Find the intercepts made by the following planes:

(a) $4x + 12y + 3z = 12$;
(b) $15x - 6y + 10z + 30 = 0$;
(c) $x - y + z - 1 = 0$;
(d) $x + 2y + 3z + 4 = 0$.

2. Interpret the following equations:

(a) $x + y + z = 1$;
(b) $5y - 3z = 12$;
(c) $x + y = 0$;
(d) $5y + 12 = 0$.

3. Find the plane determined by the points (2, 1, 3), (1, − 5, 0), (4, 6, − 1).

4. Write down the equation of the plane whose intercepts are 3, 2, − 5.

5. Find the intercepts of the plane passing through the points (3, − 1, 4), (6, 2, − 3), (− 1, − 2, − 3).

6. If planes are parallel to and a distance a from the coordinate planes, what are their intercepts? What are their equations?

7. Show that the four points (4, 3, 3), (4, − 3, − 9), (0, 0, 3), (2, 1, 2) lie in a plane and find its equation.

316. Normal Form. The position of a plane in space is fully determined by the length $p = ON$ (Fig. 133) of the perpendicular let fall from the origin on the plane and the direction cosines l, m, n of this perpendicular regarded as a vector ON. Let P be any point of the plane and $OQ = x$, $QR = y$, $RP = z$ its coordinates; as the projection of the open polygon $OQRP$ on ON is equal to ON (§ 294) we have

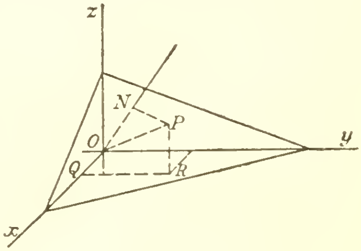


FIG. 133

$$(4) \qquad lx + my + nz = p.$$

This equation is called the *normal form* of the equation of a plane. Observe that the number p is always positive, being the distance of the plane from the origin, or the length of the vector ON. Hence $lx + my + nz$ is always positive.

317. Reduction to the Normal Form. The equation $Ax + By + Cz + D = 0$ is in general not of the form $lx + my + nz = p$ since in the latter equation the coefficients of x, y, z, being the direction cosines of a vector, have the property that the sum of their squares is equal to 1, while $A^2 + B^2 + C^2$ is in general not equal to 1. But the general equation can be reduced to the normal form by multiplying it by a constant factor k properly chosen. The equation

$$kAx + kBy + kCz + kD = 0$$

evidently represents the same plane as does the equation $Ax + By + Cz + D = 0$; and we can select k so that

$$(kA)^2 + (kB)^2 + (kC)^2 = 1, \quad \text{viz.} \quad k = \frac{1}{\pm\sqrt{A^2 + B^2 + C^2}}.$$

As in the normal form the right-hand member p is positive (§ 316) the sign of the square root should be selected so that kD becomes negative.

The normal form is therefore obtained by dividing the equation $Ax + By + Cz + D = 0$ *by* $\pm\sqrt{A^2 + B^2 + C^2}$ *according* as D *is negative or positive.*

It follows at the same time that the direction cosines of any normal to the plane $Ax + By + Cz + D = 0$ are proportional to A, B, C, viz.

$$l = \frac{A}{\pm\sqrt{A^2 + B^2 + C^2}}, \quad m = \frac{B}{\pm\sqrt{A^2 + B^2 + C^2}},$$

$$n = \frac{C}{\pm\sqrt{A^2 + B^2 + C^2}},$$

and that the distance of the plane from the origin is

$$p = \frac{-D}{\pm\sqrt{A^2 + B^2 + C^2}},$$

the upper sign of the square root to be used when D is negative, the lower when D is positive.

318. Distance of Point from Plane. Let $lx + my + nz = p$ be the equation of a plane in the normal form, $P_1(x_1, y_1, z_1)$ any point not on this plane (Fig. 134). The projection OS of the vector OP_1 on the normal to the plane being equal to the sum of the projections of its components $OQ = x_1$, $QR = y_1$, $RP_1 = z_1$, we have

$$OS = lx_1 + my_1 + nz_1.$$

Hence the distance d of P_1 from the plane, which is equal to NS, will be

$$d = OS - ON = lx_1 + my_1 + nz_1 - p.$$

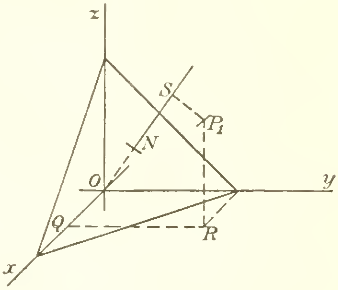


FIG. 134

If this expression is negative, the point P_1 lies on the same side of the plane as does the origin; if it is positive, the point P_1 lies on the opposite side of the plane. Any plane thus divides space into two regions, in one of which the distance of every point from the plane is positive, while in the other the distance is negative. If the plane does not pass through the origin, the region containing the origin is the negative region; if it does, either side can be taken as the positive side.

To find the distance of a point $P_1(x_1, y_1, z_1)$ from a plane given in the general form

$$Ax + By + Cz + D = 0,$$

we have only to reduce the equation to the normal form (§ 317) and then to substitute for x, y, z the coordinates x_1, y_1, z_1 of P_1; thus

$$d = \frac{Ax_1 + By_1 + Cz_1 + D}{\pm\sqrt{A^2 + B^2 + C^2}},$$

the square root being taken with + or − according as D is negative or positive.

Notice that d is the distance from the plane to the point P_1, not from P_1 to the plane.

319. Angle between Two Planes. As two intersecting planes make two angles whose sum $= \pi$, we shall, to avoid any ambiguity, define the angle between the planes as the angle between the perpendiculars (regarded as vectors) drawn from the origin to the two planes.

If the equations of the planes are given in the normal form,

$$l_1x + m_1y + n_1z = p_1,$$
$$l_2x + m_2y + n_2z = p_2,$$

we have, by § 299, for the angle ψ between the planes:

$$\cos\psi = l_1l_2 + m_1m_2 + n_1n_2.$$

If the equations of the planes are in the general form,

$$A_1x + B_1y + C_1z + D_1 = 0,$$
$$A_2x + B_2y + C_2z + D_2 = 0,$$

we find by reducing to the normal form (§ 317):

$$\cos\psi = \frac{A_1A_2 + B_1B_2 + C_1C_2}{\pm\sqrt{A_1^2 + B_1^2 + C_1^2} \cdot \pm\sqrt{A_2^2 + B_2^2 + C_2^2}}.$$

320. Bisecting Planes. To find the equations of the two planes that bisect the angles formed by two intersecting planes given in the normal form,

$$l_1x + m_1y + n_1z - p_1 = 0, \quad l_2x + m_2y + n_2z - p_2 = 0,$$

observe that for any point in either bisecting plane its distances from the two given planes must be equal in absolute value. Hence the equations of the required planes are

$$l_1x + m_1y + n_1z - p_1 = \pm(l_2x + m_2y + n_2z - p_2).$$

To distinguish the two planes, observe that for the plane that bisects that pair of vertical angles which contains the origin the perpendicular distances are in the one angle both positive, in the other both negative; hence the plus sign gives this bisecting plane.

If the equations of the planes are given in the general form,

$$A_1x + B_1y + C_1z + D_1 = 0,\quad A_2x + B_2y + C_2z + D_2 = 0,$$

first reduce the equations to the normal form (§ 317).

EXERCISES

1. A line is drawn from the origin perpendicular to the plane $x - y - 5z - 10 = 0$; what are the direction cosines of this line?

2. Find the distance from the origin to the plane $2x + 2y - z = 6$.

3. Find the distances of the following planes from the origin:

(a) $3x - 4y + 5z - 8 = 0$. (b) $x + y + z = 0$.
(c) $2y - 5z = 3$. (d) $3x - 4y + 5 = 0$.

4. Find the distances from the following planes to the point $(2, 1, -3)$:

(a) $3x + 5y - 6z = 8$. (b) $2x - 3y + z = 0$. (c) $x + y + z = 0$.

5. Find the plane through the point $(4, 8, 1)$ which is perpendicular to the radius vector of this point; also the parallel plane whose distance from the origin is 10 and in the same sense.

6. Find the plane through the point $(-1, 2, -4)$ that is parallel to the plane $4x - 3y + 2z = 8$; what is the distance between these planes?

7. Find the distance between the planes $4x - 5y - 2z = 6$, $4x - 5y - 2z + 8 = 0$.

8. Are the points $(6, 1, -4)$ and $(4, -2, 3)$ on the same side of the plane $2x - 3y - 5z + 1 = 0$?

9. Write down the equation of the plane equally inclined to the axes and at the distance p from the origin.

10. Show that the relation between the distance p from the origin to a plane and the intercepts a, b, c is $1/a^2 + 1/b^2 + 1/c^2 = 1/p^2$.

11. Show that the locus of the points equally distant from the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is a plane that bisects P_1P_2 at right angles.

12. Find the equations of the planes bisecting the angles: (a) between the planes $x + y + z - 3 = 0$, $2x - 3y + 4z + 3 = 0$; (b) between the planes $2x - 2y - z = 8$, $x + 2y + 2z = 6$.

321. Volume of a Tetrahedron. The volume of the tetrahedron whose vertices are the points $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$, $P_3(x_3, y_3, z_3)$, $P_4(x_4, y_4, z_4)$ can be expressed in terms of the coordinates of the points. The equation of the plane determined by the points P_2, P_3, P_4 is (§ 315)

$$\begin{vmatrix} x & y & z & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0.$$

Now the altitude d of the tetrahedron is the distance from this plane to the point $P_1(x_1, y_1, z_1)$, *i.e.* (§ 318)

$$d = \frac{\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}}{\sqrt{\begin{vmatrix} y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \\ y_4 & z_4 & 1 \end{vmatrix}^2 + \begin{vmatrix} z_2 & x_2 & 1 \\ z_3 & x_3 & 1 \\ z_4 & x_4 & 1 \end{vmatrix}^2 + \begin{vmatrix} x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{vmatrix}^2}}.$$

But the denominator is seen immediately to represent twice the area of the triangle with vertices P_2, P_3, P_4 (Ex. 9, p. 291), *i.e.* twice the base of the tetrahedron. Denoting the base by B, we then have

$$2\,Bd = \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}.$$

The volume of the tetrahedron is $V = \frac{1}{3}Bd$, and therefore

$$V = \frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}.$$

322. Simultaneous Linear Equations. Two simultaneous equations of the first degree,

$$A_1x + B_1y + C_1z + D_1 = 0,$$
$$A_2x + B_2y + C_2z + D_2 = 0,$$

represent in general the line of intersection of the two planes represented by the two equations separately. For, the coordinates of every point of this line, and those of no other point, satisfy both equations. See § 310 and §§ 326–327.

Three simultaneous equations of the first degree,

$$A_1x + B_1y + C_1z + D_1 = 0,$$
$$A_2x + B_2y + C_2z + D_2 = 0,$$
$$A_3x + B_3y + C_3z + D_3 = 0,$$

determine in general the point of intersection of the three planes. The coordinates of this point are found by solving the three equations for x, y, z. But it may happen that the three planes have no common point, as when the three lines of intersection are parallel, or when the three planes are parallel; and it may happen that the planes have an infinite number of points in common, as when two of the planes, or all three, coincide, or when the three planes pass through one and the same line.

Four planes will in general have no point in common. If they do, *i.e.* if there exists a point (x_1, y_1, z_1) satisfying the four equations

$$A_1x_1 + B_1y_1 + C_1z_1 + D_1 = 0,$$
$$A_2x_1 + B_2y_1 + C_2z_1 + D_2 = 0,$$
$$A_3x_1 + B_3y_1 + C_3z_1 + D_3 = 0,$$
$$A_4x_1 + B_4y_1 + C_4z_1 + D_4 = 0,$$

we can eliminate x_1, y_1, z_1, 1 between these equations so that we find the condition

$$\begin{vmatrix} A_1 & B_1 & C_1 & D_1 \\ A_2 & B_2 & C_2 & D_2 \\ A_3 & B_3 & C_3 & D_3 \\ A_4 & B_4 & C_4 & D_4 \end{vmatrix} = 0.$$

EXERCISES

1. Find the volume of the tetrahedron whose vertices are $(0, 0, 0)$, $(a, 0, 0)$, $(0, b, 0)$, $(0, 0, c)$.

2. Find the volumes of the tetrahedra whose vertices are the following points:

(*a*) $(7, 0, 6)$, $(3, 2, 1)$, $(-1, 0, 4)$, $(3, 0, -2)$.
(*b*) $(3, 0, 1)$, $(0, -8, 2)$, $(4, 2, 0)$, $(0, 0, 10)$.
(*c*) $(2, 1, -3)$, $(4, -2, 1)$, $(3, -7, -4)$, $(5, 1, 8)$.

3. Find the coordinates of the points in which the following planes intersect:

(*a*) $2x + 5y + z - 2 = 0$, $x + 5y + z = 0$, $3x - 3y + 2z - 12 = 0$.
(*b*) $2x + y + z = a + b + c$, $4x - 2y + z = 2a - 2b + c$, $6x - y = 3a - b$.

4. Show that the four planes $5x - 3y - z = 0$, $4x - 2y + z = 3$, $3x + 2y - 6z = 6$, $x + y + z = 6$ pass through the same point. What are the coordinates of this point?

5. Show that the four planes $4x + y + z + 4 = 0$, $x + 2y - z + 3 = 0$, $y - 5z + 14 = 0$, $x + y + z - 2 = 0$ have a common point.

6. Show that the locus of a point the sum of whose distances from any number of fixed planes is constant is a plane.

323. Pencil of Planes. All the planes that pass through one and the same line are said to form a *pencil* of planes, and their common line is called the ***axis*** of the pencil.

If the equations of any two non-parallel planes are given, say

$$(1) \qquad \begin{aligned} A_1x + B_1y + C_1z + D_1 &= 0, \\ A_2x + B_2y + C_2z + D_2 &= 0, \end{aligned}$$

then the equation of any other plane of the pencil having their intersection as axis can be written in the form

$$(2) \quad (A_1x + B_1y + C_1z + D_1) + k(A_2x + B_2y + C_2z + D_2) = 0,$$

where k is a constant whose value determines the position of the plane in the pencil.

For, this equation (2) being of the first degree in x, y, z certainly represents a plane; and the coordinates of the points of the line of intersection of the two given planes (1), since they satisfy each of the equations (1), must satisfy the equation (2) so that the plane (2) passes through the axis of the pencil.

324. Sheaf of Planes. All the planes that pass through one and the same point are said to form a *sheaf* of planes, and their common point is called the ***center*** of the sheaf.

If the equations of any three planes, not of the same pencil, are given, say

$$A_1x + B_1y + C_1z + D_1 = 0,$$
$$A_2x + B_2y + C_2z + D_2 = 0,$$
$$A_3x + B_3y + C_3z + D_3 = 0,$$

then the equation of any other plane of the sheaf having their point of intersection as center can be written in the form

$$(A_1x + B_1y + C_1z + D_1) + k_1(A_2x + B_2y + C_2z + D_2) \\ + k_2(A_3x + B_3y + C_3z + D_3) = 0,$$

where k_1 and k_2 are constants whose values determine the position of the plane in the sheaf.

The proof is similar to that of § 323.

325. Non-linear Equations Representing Several Planes. When two planes are given, say

$$A_1x + B_1y + C_1z + D_1 = 0,$$
$$A_2x + B_2y + C_2z + D_2 = 0,$$

then the equation

$$(A_1x + B_1y + C_1z + D_1)(A_2x + B_2y + C_2z + D_2) = 0,$$

obtained by equating to zero the product of the left-hand members (the right-hand members being reduced to zero), is satisfied by all the points of the first given plane as well as all the points of the second given plane, and by no other points.

The product equation is therefore said to represent the two given planes. The equation is of the second degree.

Similarly, by equating to zero the product of the left-hand members of the equations of three or more planes (the right-hand members being zero) we obtain a single equation representing all these planes. An equation of the nth degree *may*, therefore, represent n planes; it will do so if its left-hand member can be resolved into n linear factors with real coefficients.

EXERCISES

1. Find the plane that passes through the line of intersection of the planes $5x - 3y + 4z - 35 = 0$, $x + y - z = 0$ and through $(4, -3, 2)$.

2. Show that the planes $3x - 2y + 5z + 2 = 0$, $x + y - z - 5 = 0$, $6x + y + 2z - 13 = 0$ belong to the same pencil.

3. Show that the following planes belong to the same sheaf and find the coordinates of the center of the sheaf: $6x + y - 4z = 0$, $x + y + z = 5$, $2x - 4y - z = 10$, $2x + 3y + z = 4$.

4. What planes are represented by the following equations?
(*a*) $x^2 - 6x + 8 = 0$, (*b*) $y^2 - 9 = 0$, (*c*) $x^2 - z^2 = 0$, (*d*) $x^2 - 4xy = 0$.

5. Find the cosine of the angle between the following pairs of planes:
(*a*) $4x-3y-z=6$, $x+y-z=8$; (*b*) $2x+7y+4z=2$, $x-9y-2z=12$.

6. Show that the following pairs of planes are either parallel or perpendicular:
(*a*) $3x-2y+5z=0$, $2x+3y=8$; (*b*) $5x+2y-z=6$, $10x+4y-2z=3$;
(*c*) $x+y-2z=3$, $x+y+z=11$; (*d*) $x-2y-z=8$, $3x-6y-3z=5$.

7. Find the plane that is perpendicular to the segment joining the points $(3, -4, 6)$ and $(2, 1, -3)$ at its midpoint.

8. Show that the planes $A_1x + B_1y + C_1z + D_1 = 0$, $A_2x + B_2y + C_2z + D_2 = 0$ are parallel (on the same or opposite sides of the origin) if

$$\frac{A_1A_2 + B_1B_2 + C_1C_2}{\sqrt{A_1^2 + B_1^2 + C_1^2}\sqrt{A_2^2 + B_2^2 + C_2^2}} = \pm 1.$$

9. A cube whose edges have the length a is referred to a coordinate trihedral, the origin being taken at the center of a face and the axes parallel to the edges of the cube. Find the equations of the faces.

10. Show that the plane through the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ and perpendicular to the plane $Ax + By + Cz + D = 0$ can be represented by the equation

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ A & B & C & 0 \end{vmatrix} = 0.$$

11. Find those planes of the pencil $4x - 3y + 5z = 8$, $2x + 3y - z = 4$ which are perpendicular to the coordinate planes.

12. Find the plane that is perpendicular to the plane $2x + 3y - z = 1$ and passes through the points $(1, 1, -1)$, $(3, 4, 2)$.

13. Find the plane that is perpendicular to the planes $4x - 3y + z = 6$, $2x + 3y - 5z = 4$ and passes through the point $(4, -1, 5)$.

14. Show that the conditions that three planes $A_1x + B_1y + C_1z + D_1 = 0$, $A_2x + B_2y + C_2z + D_2 = 0$, $A_3x + B_3y + C_3z + D_3 = 0$ belong to the same pencil, are

$$\frac{A_1 + k\,A_2}{A_3} = \frac{B_1 + k\,B_2}{B_3} = \frac{C_1 + k\,C_2}{C_3} = \frac{D_1 + k\,D_2}{D_3};$$

or, putting these fractions equal to s and eliminating k and s,

$$\begin{vmatrix} B_1 & C_1 & D_1 \\ B_2 & C_2 & D_2 \\ B_3 & C_3 & D_3 \end{vmatrix} = \begin{vmatrix} C_1 & D_1 & A_1 \\ C_2 & D_2 & A_2 \\ C_3 & D_3 & A_3 \end{vmatrix} = \begin{vmatrix} D_1 & A_1 & B_1 \\ D_2 & A_2 & B_2 \\ D_3 & A_3 & B_3 \end{vmatrix} = \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = 0.$$

(Verify Ex. 2 by using these conditions.)

15. Find the equations of the faces of a right pyramid, with square base of side $2a$ and with altitude h, the origin being taken at the center of the base, the axis Oz through the opposite vertex and the axes Ox, Oy parallel to the sides of the base.

16. Homogeneous substances passing from a liquid to a solid state tend to form crystals; *e.g.* an ideal specimen of ammonium alum has the form of a regular octahedron. Find the equations of the faces of such a crystal of edge a if the origin is taken at the center and the axes through the vertices, and determine the angle between two faces.

17. Find the angles between the lateral faces of a right pyramid whose base is a regular hexagon of side a and whose altitude is h.

PART II. THE STRAIGHT LINE

326. Determination of Direction Cosines. Two simultaneous linear equations (§ 322),

$$(1) \qquad Ax+By+Cz+D=0, \quad A'x+B'y+C'z+D'=0,$$

represent a line, namely, the intersection of the two planes represented by the two equations separately, provided the two planes are not parallel.

To obtain the direction cosines l, m, n of this line observe that the line, since it lies in each of the two planes, is perpendicular to the normal of each plane. Now, by § 317 the direction cosines of these normals are proportional to A, B, C and A', B', C', respectively. We have therefore

$$Al+Bm+Cn=0, \quad A'l+B'm+C'n=0,$$

whence

$$l:m:n=\begin{vmatrix} B & C \\ B' & C' \end{vmatrix} : \begin{vmatrix} C & A \\ C' & A' \end{vmatrix} : \begin{vmatrix} A & B \\ A' & B' \end{vmatrix}.$$

The direction cosines themselves are then found by dividing each of these determinants by the square root of the sum of their squares.

327. Intersecting Lines. The two lines

$$\left.\begin{array}{l} A_1x+B_1y+C_1z+D_1=0, \\ A_1'x+B_1'y+C_1'z+D_1'=0 \end{array}\right\} \text{ and } \left\{\begin{array}{l} A_2x+B_2y+C_2z+D_2=0, \\ A_2'x+B_2'y+C_2'z+D_2'=0 \end{array}\right.$$

will intersect if, and only if, the four planes represented by these equations have a common point. By § 322, the condition for this is

$$\begin{vmatrix} A_1 & B_1 & C_1 & D_1 \\ A_1' & B_1' & C_1' & D_1' \\ A_2 & B_2 & C_2 & D_2 \\ A_2' & B_2' & C_2' & D_2' \end{vmatrix}=0.$$

328. Special Forms of Equations. For many purposes it is convenient to represent a line by means of one of its points and its direction cosines, or by means of two of its points. Let the line be called λ.

If (x_1, y_1, z_1) is a given point of λ and l, m, n are the direction cosines of λ, then every point (x, y, z) of λ must satisfy the relations (§ 298):

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}. \tag{2}$$

In these equations, l, m, n, *can evidently be replaced by any three numbers proportional to* l, m, n. Thus, if (x_2, y_2, z_2) be any point of λ different from (x_1, y_1, z_1), we have the continued proportion

$$x_2-x_1 : y_2-y_1 : z_2-z_1 = l : m : n;$$

hence *the equations of the line through the two points* (x_1, y_1, z_1) *and* (x_2, y_2, z_2) are:

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}. \tag{3}$$

If, for the sake of brevity, we put $x_2-x_1=a$, $y_2-y_1=b$, $z_2-z_1=c$, we can write the equations of the line in the form

$$\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}, \tag{4}$$

where a, b, c, are proportional to l, m, n, and can be regarded as the components of a vector parallel to the line.

The equations (3) also follow directly by eliminating k between the equations of § 295, namely,

$$x=x_1+k(x_2-x_1),\ y=y_1+k(y_2-y_1),\ z=z_1+k(z_2-z_1). \tag{5}$$

These equations which, with a variable k, represent any point of the line through (x_1, y_1, z_1) and (x_2, y_2, z_2) are called the ***parameter equations*** of the line.

329. Projecting Planes of a Line. Each of the forms (2), (3), (4), which are not essentially different, furnishes three linear equations; thus (4) gives:

$$\frac{y-y_1}{b}=\frac{z-z_1}{c}, \qquad \frac{z-z_1}{c}=\frac{x-x_1}{a}, \qquad \frac{x-x_1}{a}=\frac{y-y_1}{b};$$

but these three equations are equivalent to only two, since from any two the third follows immediately. The first of these equations, which can be written in the form

$$cy-bz-(cy_1-bz_1)=0,$$

represents, since it does not contain x (§ 313), a plane perpendicular to the plane Oyz; and as this plane must contain the line λ it is *the plane CC′A that projects* λ *on the plane* Oyz (Fig. 135). Similarly the other two equations represent the planes that project λ on the coordinate planes Ozx and Oxy. Any two of these equations represent the line λ as the intersection of two of these projecting planes.

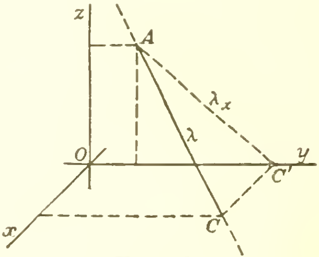


Fig. 135

At the same time the equation

$$\frac{y-y_1}{b}=\frac{z-z_1}{c}$$

can be interpreted as representing a line in the plane Oyz, viz. the intersection of the projecting plane with the plane $x=0$. This line (AC' in Fig. 135) is *the projection* λ_x *of* λ *on the plane* Oyz. As the other two equations (4) can be interpreted similarly it appears that the equations (2), (3), or (4) represent the line λ by means of its projections λ_x, λ_y, λ_z on the three coordinate planes, just as is done in descriptive geometry. Any two of the projections are of course sufficient to determine the line.

330. Determination of Projecting Planes. To reduce the equations of a line λ given in the form (1) to the form (4) we have only to eliminate between the equations (1) first one of the variables x, y, z, then another, so as to obtain two equations, each in only two variables (not the same in both).

The process will best be understood from an example. The line being given as the intersection of the planes

$$(a)\ 2x - 3y + z + 3 = 0,$$
$$(b)\ x + y + z - 2 = 0,$$

eliminate z by subtracting (b) from (a) and eliminate x by subtracting (b), multiplied by 2, from (a); this gives the line as the intersection of the planes

$$x - 4y + 5 = 0,$$
$$-5y - z + 7 = 0,$$

which are the projecting planes parallel to Oz and Ox, *i.e.* the planes that project the line on Oxy and Oyz. Solving for y and equating the two values of y we find:

$$\frac{x+5}{4} = \frac{y}{1} = \frac{z-7}{-5}.$$

The line passes therefore through the point $(-5, 0, 7)$ and has direction cosines proportional to $4, 1, -5$, viz.

$$l = \frac{4}{\sqrt{42}}, \qquad m = \frac{1}{\sqrt{42}}, \qquad n = -\frac{5}{\sqrt{42}}.$$

EXERCISES

1. Write the equations of the line through the point $(-3, 1, 6)$ whose direction cosines are proportional to $3, 5, 7$.

2. Write the equations of the line through the point $(3, 2 -4)$ whose direction cosines are proportional to $5, -1, 3$.

3. Find the line through the point (a, b, c) that is equally inclined to the axes of coordinates.

4. Find the lines that pass through the following pairs of points:
(*a*) $(4, -3, 1)$, $(2, 3, 2)$, (*b*) $(-1, 2, 3)$, $(8, 7, 1)$,
(*c*) $(-2, 3, -4)$, $(0, 2, 0)$, (*d*) $(-1, -5, -2)$, $(-3, 0, -1)$,
and determine the direction cosines of each of these lines.

5. Find the traces of the plane $2x - 3y - 4z = 6$ in the coordinate planes.

6. Write the equations of the line $2x-3y+5z-6=0$, $x-y+2z-3=0$ in the form (4) and determine the direction cosines.

7. Put the line $4x - 3y - 6 = 0$, $x - y - z - 4 = 0$ in the form (4) and determine the direction cosines.

8. Find the line through the point $(2, 1, -3)$ that is parallel to the line $2x - 3y + 4z - 6 = 0$, $5x + y - 2z - 8 = 0$.

9. What are the projections of the line $5x - 3y - 7z - 10 = 0$, $x + y - 3z + 5 = 0$ on the coordinate planes?

10. Obtain the equations of the line through two given points by equating the values of k obtained from § 295.

11. By § 317, the direction cosines of any line are proportional to the coefficients of x, y, and z in the equation of a plane perpendicular to the line. Find a line through the point $(3, 5, 8)$ that is perpendicular to the plane $2x + y + 3z = 5$.

331. Angle between Two Lines. The cosine of the angle ψ between two lines whose direction cosines are l_1, m_1, n_1 and l_2, m_2, n_2 is, by § 299,

$$\cos\psi = l_1l_2 + m_1m_2 + n_1n_2.$$

Hence if the lines are given in the form (4), say

$$\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}, \quad \frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2},$$

we have

$$\cos\psi = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\pm\sqrt{a_1^2 + b_1^2 + c_1^2} \cdot \pm\sqrt{a_2^2 + b_2^2 + c_2^2}}.$$

If the lines are *parallel*, then

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2};$$

if they are *perpendicular*, then

$$a_1a_2 + b_1b_2 + c_1c_2 = 0;$$

and *vice versa.*

332. Angle between Line and Plane. Let the line and plane be given by the equations

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c},$$

$$Ax + By + Cz + D = 0.$$

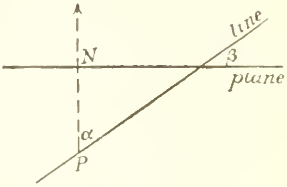


FIG. 136

The plane of Fig. 136 represents the plane through the given line perpendicular to the given plane. The angle β between the given line and plane is the complement of the angle α between the line and any perpendicular PN to the plane. Hence

$$\sin \beta = \frac{aA + bB + cC}{\pm\sqrt{a^2 + b^2 + c^2} \cdot \pm\sqrt{A^2 + B^2 + C^2}}.$$

The (necessary and sufficient) condition for *parallelism* of line and plane is

$$aA + bB + cC = 0;$$

the condition of *perpendicularity* is

$$\frac{a}{A} = \frac{b}{B} = \frac{c}{C}.$$

333. Line and Plane Perpendicular at Given Point. If the plane

$$Ax + By + Cz + D = 0$$

passes through the point $P_1(x_1, y_1, z_1)$, we must have

$$Ax_1 + By_1 + Cz_1 + D = 0.$$

Subtracting from the preceding equation, we have as *the equation of any plane through the point* $P_1(x_1, y_1, z_1)$:

$$A(x - x_1) + B(y - y_1) + C(z - z_1) = 0.$$

The equations of any line through the same point are

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}.$$

If this line is perpendicular to the plane, we must have (§ 332): $a/A = b/B = c/C$. Hence the equations

$$\frac{x - x_1}{A} = \frac{y - y_1}{B} = \frac{z - z_1}{C}$$

represent the line through $P_1(x_1, y_1, z_1)$ perpendicular to the plane $A(x - x_1) + B(y - y_1) + C(z - z_1) = 0$.

omit

334. Distance of a Point from a Line. If the equations of the line λ are given in the form

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n},$$

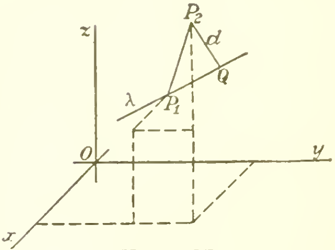


FIG. 137

where (x_1, y_1, z_1) is a point P_1 of λ (Fig. 137), the distance $d = QP_2$ of the point $P_2(x_2, y_2, z_2)$ from λ can be found from the right-angled triangle P_1QP_2 which gives

$$d^2 = P_1P_2^2 - P_1Q^2,$$

by observing that

$$P_1P_2^2 = (x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2,$$

while P_1Q is the projection of P_1P_2 on λ. This projection is found (§ 294) as the sum of the projections of the components $x_2 - x_1$, $y_2 - y_1$, $z_2 - z_1$ of P_1P_2 on λ:

$$P_1Q = l(x_2-x_1) + m(y_2-y_1) + n(z_2-z_1).$$

Hence

$$d^2=(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2-[l(x_2-x_1)+m(y_2-y_1)+n(z_2-z_1)]^2.$$

335. Shortest Distance between Two Lines. *Two lines* λ_1, λ_2 whose equations are given in the form

$$\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1},\quad \frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2}$$

will intersect if their directions (l_1, m_1, n_1), (l_2, m_2, n_2), and the direction of the line joining the points (x_1, y_1, z_1), (x_2, y_2, z_2) are complanar (§ 302), *i.e.* if

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0.$$

If the lines λ_1, λ_2 do not intersect, their shortest distance d is the distance of $P_2(x_2, y_2, z_2)$ from the plane through λ_1 parallel to λ_2. As this plane contains the directions of λ_1 and λ_2, the direction cosines of its normal are (§ 301) proportional to

$$\begin{vmatrix} m_1 & n_1 \\ m_2 & n_2 \end{vmatrix},\quad \begin{vmatrix} n_1 & l_1 \\ n_2 & l_2 \end{vmatrix},\quad \begin{vmatrix} l_1 & m_1 \\ l_2 & m_2 \end{vmatrix};$$

and as it passes through $P_1(x_1, y_1, z_1)$ its equation can be written in the form

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0.$$

Hence the ***shortest distance of the lines*** λ_1, λ_2 is:

$$d = \frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}}{\sqrt{\begin{vmatrix} m_1 & n_1 \\ m_2 & n_2 \end{vmatrix}^2 + \begin{vmatrix} n_1 & l_1 \\ n_2 & l_2 \end{vmatrix}^2 + \begin{vmatrix} l_1 & m_1 \\ l_2 & m_2 \end{vmatrix}^2}}.$$

As the denominator of this expression is equal to $\sin\psi$ (§ 301), we have

$$d \sin\psi = \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}.$$

EXERCISES

1. Find the cosine of the angle between the lines

$$\frac{x-3}{2} = \frac{y-5}{3} = \frac{z+1}{4} \quad \text{and} \quad \frac{x+1}{-1} = \frac{y-3}{2} = \frac{z+3}{3}.$$

2. Find the angle between the lines $3x - 2y + 4z - 1 = 0$, $2x + y - 3z + 10 = 0$, and $x + y + z = 6$, $2x + 3y - 5z = 8$.

3. Find the angle between the lines that pass through the points $(4, 2, 5)$, $(-2, 4, 3)$ and $(-1, 4, 2)$, $(4, -2, -6)$.

4. Find the angle between the line

$$\frac{x+1}{3} = \frac{y-2}{-5} = \frac{z+10}{3}$$

and a perpendicular to the plane $4x - 3y - 2z = 8$.

5. In what ratio does the plane $3x - 4y + 6z - 8 = 0$ divide the segment drawn from the origin to the point $(10, -8, 4)$.

6. Find the plane through the point $(2, -1, 3)$ perpendicular to the line

$$\frac{x-3}{4} = \frac{y+2}{3} = \frac{z-7}{-1}.$$

7. Find the plane that is perpendicular to the line $4x + y - z = 6$, $3x + 4y + 8z + 10 = 0$ and passes through the point $(4, -1, 3)$.

8. Find the plane through the origin perpendicular to the line

$$5x - 2y + z = 6, \quad 3x + y - 4z = 8.$$

9. Find the plane through the point $(4, -3, 1)$ perpendicular to the line joining the points $(3, 1, -6)$, $(-2, 4, 7)$.

10. Find the line through the point $(2, -1, 4)$ perpendicular to the plane $x - 2y + 4z = 6$.

11. Show that the lines $x/3 = y/-1 = z/-2$ and $x/4 = y/6 = z/3$ are perpendicular.

12. Show that the lines

$$\frac{x-1}{1} = \frac{y+2}{-2} = \frac{z-3}{3} \quad \text{and} \quad \frac{x-2}{-2} = \frac{y-3}{4} = \frac{z}{-6}$$

are parallel.

13. Find the angle between the line $3x - 2y - z = 4$, $4x + 3y - 3z = 6$ and the plane $x + y + z = 8$.

14. Find the lines bisecting the angles between the lines

$$\frac{x-a}{l_1} = \frac{y-b}{m_1} = \frac{z-c}{n_1} \quad \text{and} \quad \frac{x-a}{l_2} = \frac{y-b}{m_2} = \frac{z-c}{n_2}.$$

15. Find the plane perpendicular to the plane $3x - 4y - z = 6$ and passing through the points $(1, 3, -2)$, $(2, 1, 4)$.

16. Find the plane through the point $(3, -1, 2)$ perpendicular to the line $2x - 3y - 4z = 7$, $x + y - 2z = 4$.

17. Find the plane through the point (a, b, c) perpendicular to the line $A_1x + B_1y + C_1z + D_1 = 0$, $A_2x + B_2y + C_2z + D_2 = 0$.

18. Find the projection of the vector from $(3, 4, 5)$ to $(2, -1, 4)$ on the line that makes equal angles with the axes; and on the plane

$$2x - 3y + 4z = 6.$$

19. Find the distances from the following lines to the points indicated:

(*a*) $\dfrac{x}{3} = \dfrac{y-2}{5} = \dfrac{z+1}{2}$, $(0, 0, 0)$;

(*b*) $2x + y - z = 6$, $x - y + 4z = 8$, $(3, 1, 4)$;

(*c*) $2x + 3y + 5z = 1$, $3x - 6y + 3z = 0$, $(4, 1, -2)$.

20. Show that the equation of the plane determined by the line

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

and the point $P_2(x_2, y_2, z_2)$ can be written in the form

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a & b & c \end{vmatrix} = 0.$$

21. Find the plane determined by the intersecting lines

$$\frac{x-3}{4} = \frac{y-5}{3} = \frac{z+1}{2} \text{ and } \frac{x-3}{1} = \frac{y-5}{2} = \frac{z+1}{3}.$$

22. Find the plane determined by the line

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c},$$

and its parallel through the point $P_2(x_2, y_2, z_2)$.

23. Given two non-intersecting lines

$$\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}, \quad \frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2};$$

find the plane passing through the first line and a parallel to the second; and the plane passing through the second line and a parallel to the first.

24. What is the condition that the two lines of Ex. 23 intersect?

25. Find the distance from the diagonal of a cube to a vertex not on the diagonal.

26. Find the distance between the lines given in Ex. 23.

27. Show that the locus of the points whose distances from two fixed planes are in constant ratio is a plane.

28. Show that the plane $(m - n)x + (n - l)y + (l - m)z = 0$ contains the line $x/l = y/m = z/n$ and is perpendicular to the plane determined by the lines $x/m = y/n = z/l$ and $x/n = y/l = z/m$.

CHAPTER XV

THE SPHERE

336. Spheres. A sphere is defined as the locus of all those points that have the same distance from a fixed point.

Let $C(h, j, k)$ denote the center, and r the radius, of a sphere; the necessary and sufficient condition that any point $P(x, y, z)$ has the distance r from $C(h, j, k)$ is

$$(1) \qquad (x-h)^2+(y-j)^2+(z-k)^2=r^2.$$

This then is *the cartesian equation of the sphere of center* $C(h, j, k)$ *and radius* r.

If the center of the sphere lies in the plane Oxy, the equation becomes

$$(x-h)^2+(y-j)^2+z^2=r^2.$$

If the center lies on the axis Ox, the equation is

$$(x-h)^2+y^2+z^2=r^2.$$

The equation of a sphere about the origin as center is:

$$x^2+y^2+z^2=r^2.$$

337. Expanded Form. Expanding the squares in the equation (1), we find the equation of the sphere in the form

$$x^2+y^2+z^2-2\,hx-2\,jy-2\,kz+h^2+j^2+k^2-r^2=0.$$

This is an equation of the second degree in x, y, z; but it is of a particular form.

The ***general equation of the second degree*** in x, y, z is

$$Ax^2+By^2+Cz^2+2\,Dyz+2\,Ezx+2\,Fxy$$
$$+2\,Gx+2\,Hy+2\,Iz+J=0;$$

i.e. it contains a constant term J; three terms of the first degree, one in x, one in y, and one in z; and six terms of the second degree, one each in x^2, y^2, z^2, yz, zx, and xy.

If in the general equation we have

$$D = E = F = 0, \quad A = B = C \neq 0,$$

it reduces, upon division by A, to the form

$$x^2 + y^2 + z^2 + \frac{2\,G}{A}x + \frac{2\,H}{A}y + \frac{2\,I}{A}z + \frac{J}{A} = 0,$$

which agrees with the above form of the equation of a sphere, apart from the notation for the coefficients.

338. Determination of Center and Radius. To determine the locus represented by the equation

$$(2) \qquad Ax^2 + Ay^2 + Az^2 + 2\,Gx + 2\,Hy + 2\,Iz + J = 0,$$

where A, G, H, I, J, are any real numbers while $A \neq 0$, we divide by A and complete the squares in x, y, z; this gives

$$\left(x + \frac{G}{A}\right)^2 + \left(y + \frac{H}{A}\right)^2 + \left(z + \frac{I}{A}\right)^2 = \frac{G^2}{A^2} + \frac{H^2}{A^2} + \frac{I^2}{A^2} - \frac{J}{A}.$$

The left side represents the square of the distance of the point (x, y, z) from the point $(-G/A, -H/A, -I/A)$; the right side is constant. Hence, if the right side is positive, the equation represents the sphere whose center has the coordinates

$$h = -\frac{G}{A}, \quad j = -\frac{H}{A}, \quad k = -\frac{I}{A},$$

and whose radius is

$$r = \frac{1}{A}\sqrt{G^2 + H^2 + I^2 - AJ}.$$

If, however, $G^2 + H^2 + I^2 < AJ$, the equation is not satisfied by any point with real coordinates. If $G^2 + H^2 + I^2 = AJ$, the equation is satisfied only by the coordinates of the point $(-G/A, -H/A, -I/A)$.

Thus *the equation of the second degree*

$$Ax^2 + By^2 + Cz^2 + 2\,Dyz + 2\,Ezx + 2\,Fxy \\ + 2\,Gx + 2\,Hy + 2\,Iz + J = 0,$$

represents a sphere if, and only if,

$$A = B = C \neq 0, \quad D = E = F = 0, \quad G^2 + H^2 + I^2 > AJ.$$

339. Essential Constants. The equation (1) of the sphere contains four constants: h, j, k, r. The equation (2) contains five constants of which, however, only four are essential since we can divide out by one of these constants. Thus dividing by A and putting $2\,G/A = a$, $2\,H/A = b$, $2\,I/A = c$, $J/A = d$, the general equation (2) assumes the form

$$x^2 + y^2 + z^2 + ax + by + cz + d = 0,$$

with only the four essential constants a, b, c, d.

This fact corresponds to the possibility of determining a sphere geometrically, in a variety of ways, by four conditions.

340. Sphere through Four Points. *To find the equation of the sphere passing through four points* $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$, $P_3(x_3, y_3, z_3)$, $P_4(x_4, y_4, z_4)$, observe that the coordinates of these points must satisfy the equation of the sphere

$$x^2 + y^2 + z^2 + ax + by + cz + d = 0;$$

i.e. we must have

$$\begin{aligned} x_1^2 + y_1^2 + z_1^2 + ax_1 + by_1 + cz_1 + d &= 0,\\ x_2^2 + y_2^2 + z_2^2 + ax_2 + by_2 + cz_2 + d &= 0,\\ x_3^2 + y_3^2 + z_3^2 + ax_3 + by_3 + cz_3 + d &= 0,\\ x_4^2 + y_4^2 + z_4^2 + ax_4 + by_4 + cz_4 + d &= 0. \end{aligned}$$

As these five equations are linear and homogeneous in 1, a, b, c, d, we can eliminate these five quantities by placing the determinant of their coefficients equal to zero. Hence the equation of the desired sphere is

$$\begin{vmatrix} x^2 + y^2 + z^2 & x & y & z & \\ x_1^2 + y_1^2 + z_1^2 & x_1 & y_1 & z_1 & 1 \\ x_2^2 + y_2^2 + z_2^2 & x_2 & y_2 & z_2 & 1 \\ x_3^2 + y_3^2 + z_3^2 & x_3 & y_3 & z_3 & 1 \\ x_4^2 + y_4^2 + z_4^2 & x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0.$$

EXERCISES

1. Find the spheres with the following points as centers and with the indicated radii:

(*a*) $(4, -1, 2)$, 4; (*b*) $(0, 0, 4)$, 4; (*c*) $(2, -2, 1)$, 3; (*d*) $(3, 4, 1)$, 7.

2. Find the following spheres:

(*a*) with the points $(4, 2, 1)$ and $(3, -7, 4)$ as ends of a diameter;

(*b*) tangent to the coordinate planes and of radius a;

(*c*) with center at the point $(4, 1, 5)$ and passing through $(8, 3, -5)$.

3. Find the centers and the radii of the following spheres:

(*a*) $x^2 + y^2 + z^2 - 3x + 5y - 6z + 2 = 0$.

(*b*) $x^2 + y^2 + z^2 - 2bx + 2cz - b^2 - c^2 = 0$.

(*c*) $2x^2 + 2y^2 + 2z^2 + 3x - y + 5z - 11 = 0$.

(*d*) $x^2 + y^2 + z^2 - x - y - z = 0$.

4. Show that the equation $A(x^2 + y^2 + z^2) + 2Gx + 2Hy + 2Iz + J = 0$, in which J is variable, represents a family of concentric spheres.

5. Find the spheres that pass through the following points:

(*a*) $(1, 1, 1)$, $(3, -1, 4)$, $(-1, 2, 1)$, $(0, 1, 0)$.

(*b*) $(0, 0, 0)$, $(a, 0, 0)$, $(0, b, 0)$, $(0, 0, c)$.

(*c*) $(0, 0, 0)$, $(-1, 1, 0)$, $(1, 0, 2)$, $(0, 1, -1)$.

(*d*) $(0, 0, 0)$, $(0, 0, 4)$, $(3, 3, 3)$, $(0, 4, 0)$.

6. Find the center and radius of the sphere that is the locus of the points three times as far from the point (a, b, c) as from the origin.

7. Show that the locus of the points, the ratio of whose distances from two given points is constant, is a sphere except when the ratio is unity.

8. Find the positions of the following points relative to the sphere $x^2 + y^2 + z^2 - 4x + 4y - 2z = 0$; (*a*) the origin, (*b*) $(2, -2, 1)$, (*c*) $(1, 1, 1)$, (*d*) $(3, -2, 1)$.

9. Find the positions of the following planes relative to the sphere

$$x^2 + y^2 + z^2 + 4x - 3y + 6z + 5 = 0:$$

(*a*) $4x + 2y + z + 2 = 0$, (*b*) $8x - y - 4z + 5 = 0$.

10. Find the positions of the following lines relative to the sphere of Ex. 9:

(*a*) $2x - y + 2z + 7 = 0$, $3x - y - z - 10 = 0$.

(*b*) $3x + 8y + z - 9 = 0$, $x - 8y + z + 11 = 0$.

11. Find the coordinates of the ends of that diameter of the sphere $x^2 + y^2 + z^2 - 6x - 6y + 4z - 66 = 0$, which lies on the line joining the origin and the center.

341. Equations of a Circle. In solid analytic geometry a curve is represented by two simultaneous equations (§ 310), that is, by the equations of any two surfaces intersecting in the curve. Thus two *linear* equations represent together the line of intersection of the two planes represented by the two equations taken separately (§§ 322, 326).

A linear equation together with the equation of a sphere,

$$(3) \qquad \begin{aligned} Ax + By + Cz + D &= 0, \\ x^2 + y^2 + z^2 + ax + by + cz + d &= 0, \end{aligned}$$

represents the locus of all those points, and only those points, which the plane and sphere have in common. Thus, if the plane intersects the sphere, these simultaneous equations represent the *circle* in which the plane cuts the sphere; if the plane is tangent to the sphere, the equations represent the point of contact; if the plane does not intersect or touch the sphere, the equations are not satisfied simultaneously by any real point.

342. Sections Perpendicular to Axes. Projecting Cylinders. In particular, the simultaneous equations

$$(4) \qquad z = k, \qquad x^2 + y^2 + z^2 = r^2$$

represent, if $k < r$, a *circle about the axis* Oz (*i.e.* a circle whose center lies on Oz and whose plane is perpendicular to Oz). If the value of z obtained from the linear equation be substituted in the equation of the sphere, we obtain an equation in x and y, viz.

$$x^2 + y^2 = r^2 - k^2,$$

which represents (since z is arbitrary) the circular cylinder, about Oz as axis, which projects the circle (4) on the plane Oxy. Interpreted in the plane Oxy, *i.e.* taken together with $z = 0$, this equation represents the projection of the circle (4) on the plane Oxy.

Similarly if we eliminate x or y or z between the equations

(3) we obtain an equation in y and z, z and x, or x and y, representing the cylinder that projects the circle (3) on the plane Oyz, Ozx, or Oxy, respectively.

343. Tangent Plane. The tangent plane to a sphere at any point P_1 of the sphere is the plane through P_1, at right angles to the radius through P_1.

For a sphere whose center is at the origin,

$$x^2 + y^2 + z^2 = r^2,$$

the equation of the tangent plane at $P_1(x_1, y_1, z_1)$ is found by observing that its distance from the origin is r and that the direction cosines of its normal are those of OP_1, viz. x_1/r, y_1/r, z_1/r. Hence the equation

$$x_1x + y_1y + z_1z = r^2. \tag{5}$$

If the equation of the sphere is given in the general form

$$A(x^2 + y^2 + z^2) + 2\,Gx + 2\,Hy + 2\,Iz + J = 0,$$

we obtain by transforming to parallel axes through the center the equation

$$x^2 + y^2 + z^2 = \frac{G^2}{A^2} + \frac{H^2}{A^2} + \frac{I^2}{A^2} - \frac{J}{A} = r^2;$$

the tangent plane at $P_1(x_1, y_1, z_1)$ then is

$$x_1x + y_1y + z_1z = \frac{G^2}{A^2} + \frac{H^2}{A^2} + \frac{I^2}{A^2} - \frac{J}{A}.$$

Transforming back to the original axes, we have:

$$\left(x_1 + \frac{G}{A}\right)\left(x + \frac{G}{A}\right) + \left(y_1 + \frac{H}{A}\right)\left(y + \frac{H}{A}\right) + \left(z_1 + \frac{I}{A}\right)\left(z + \frac{I}{A}\right)$$
$$= \frac{G^2}{A^2} + \frac{H^2}{A^2} + \frac{I^2}{A^2} - \frac{J}{A}.$$

Multiplying out and rearranging, we find that *the equation of the tangent plane to the sphere*

$$A\,(x^2 + y^2 + z^2) + 2\,Gx + 2\,Hy + 2\,Iz + J = 0$$

at the point $P_1\,(x_1, y_1, z_1)$ is

$$A(x_1x + y_1y + z_1z) + G(x_1 + x) + H(y_1 + y) + I(z_1 + z) + J = 0. \tag{6}$$

344. Intersection of Line and Sphere. The intersections of a sphere about the origin,

$$x^2 + y^2 + z^2 = r^2,$$

with a line determined by two of its points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$, and given in the parameter form [(5), § 328]

$$x = x_1 + k(x_2 - x_1),\ \ y = y_1 + k(y_2 - y_1),\ \ z = z_1 + k(z_2 - z_1),$$

are found by substituting these values of x, y, z in the equation of the sphere and solving the resulting quadratic equation in k:

$$[x_1 + k(x_2 - x_1)]^2 + [y_1 + k(y_2 - y_1)]^2 + [z_1 + k(z_2 - z_1)]^2 = r^2,$$

which takes the form

$$[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]k^2 + 2[x_1(x_2 - x_1) + y_1(y_2 - y_1) + z_1(z_2 - z_1)]k + (x_1^2 + y_1^2 + z_1^2 - r^2) = 0.$$

The line P_1P_2 will intersect the sphere in two different points, be tangent to the sphere, or not meet it at all, according as the roots of this equation in k are real and different, real and equal, or imaginary; *i.e.* according as

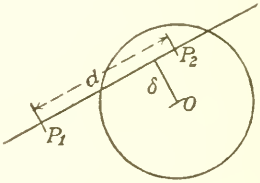


FIG. 138

$$[x_1(x_2-x_1)+y_1(y_2-y_1)+z_1(z_2-z_1)]^2-d^2(x_1^2+y_1^2+z_1^2)+d^2r^2 \gtreqless 0,$$

where d denotes the distance of the points P_1 and P_2. Dividing by d^2, we can write this condition in the form

$$r^2 - \left[x_1^2 + y_1^2 + z_1^2 - \left(x_1\frac{x_2 - x_1}{d} + y_1\frac{y_2 - y_1}{d} + z_1\frac{z_2 - z_1}{d}\right)^2\right] \gtreqless 0,$$

where by § 334 the quantity in square brackets is the square of the distance δ from the line P_1P_2 to the origin O (Fig. 138). Our condition means therefore that the line P_1P_2 meets the sphere in two different points, touches it, or does not meet it at all according as

$$r \gtreqless \delta,$$

which is obvious geometrically.

345. Tangent Cone. The condition for the line P_1P_2 to be tangent to the sphere is (§ 344):

$$[x_1(x_2-x_1)+y_1(y_2-y_1)+z_1(z_2-z_1)]^2 = (x_1^2+y_1^2+z_1^2-r^2)[(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2].$$

To give this expression a more symmetric form let us put, to abbreviate,

$$x_1x_2+y_1y_2+z_1z_2=p, \qquad x_1^2+y_1^2+z_1^2=q_1, \qquad x_2^2+y_2^2+z_2^2=q_2,$$

so that the condition is

$$(p-q_1)^2=(q_1-r^2)(q_1-2p+q_2),$$

i.e.

$$p^2-2r^2p=q_1q_2-r^2q_1-r^2q_2;$$

adding r^4 in both members, we have

$$(p-r^2)^2=(q_1-r^2)(q_2-r^2),$$

i.e.

$$(x_1x_2+y_1y_2+z_1z_2-r^2)^2=(x_1^2+y_1^2+z_1^2-r^2)(x_2^2+y_2^2+z_2^2-r^2).$$

Now keeping the sphere and the point P_1 fixed, let P_2 vary subject only to this condition, *i.e.* to the condition that P_1P_2 shall be tangent to the sphere; the point P_2, which we shall now call $P(x, y, z)$ is then any point of the cone of vertex P_1 tangent to the cone. Hence *the equation of the cone of vertex* $P_1(x_1, y_1, z_1)$ *tangent to the sphere* $x^2+y^2+z^2=r^2$ is

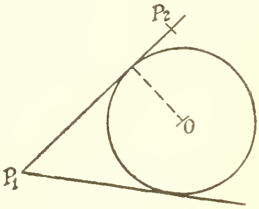


FIG. 139

$$(x_1^2+y_1^2+z_1^2-r^2)(x^2+y^2+z^2-r^2)=(x_1x+y_1y+z_1z-r^2)^2.$$

If, in particular, the point P_1 is taken on the sphere so that $x_1^2+y_1^2+z_1^2=r^2$, the equation of the tangent cone reduces to the form

$$x_1x+y_1y+z_1z=r^2,$$

which represents the tangent plane at P_1.

346. Inversion. A sphere of center O and radius a being given, we can find to every point P of space (excepting O) one and only one point P' on OP (produced if necessary) such that

$$OP \cdot OP' = a^2.$$

The points P, P' are said to be *inverse* to each other with respect to the sphere (compare § 91).

Taking rectangular axes through O, we find as the relations between the coordinates of the two inverse points $P(x, y, z)$ and $P'(x', y', z')$ if we put $OP = r = \sqrt{x^2 + y^2 + z^2}$, $OP' = r' = \sqrt{x'^2 + y'^2 + z'^2}$:

$$\frac{x'}{x} = \frac{y'}{y} = \frac{z'}{z} = \frac{r'}{r} = \frac{rr'}{r^2} = \frac{a^2}{r^2};$$

hence $$x' = \frac{a^2x}{x^2 + y^2 + z^2}, \qquad y' = \frac{a^2y}{x^2 + y^2 + z^2}, \qquad z' = \frac{a^2z}{x^2 + y^2 + z^2};$$

and similarly

$$x = \frac{a^2x'}{x'^2 + y'^2 + z'^2}, \qquad y = \frac{a^2y'}{x'^2 + y'^2 + z'^2}, \qquad z = \frac{a^2z'}{x'^2 + y'^2 + z'^2}.$$

These equations enable us to find to any surface whose equation is given the equation of the inverse surface, by simply substituting for x, y, z their values.

Thus it can be shown, that by inversion every sphere is transformed into a sphere or a plane. The proof is similar to the corresponding proposition in plane analytic geometry (§ 92) and is left as an exercise.

EXERCISES

1. Find the radius of the circle which is the intersection: (a) of the plane $y = 6$ with the sphere $x^2 + y^2 + z^2 - 6y = 0$; (b) of the plane $2x - 3y + z - 2 = 0$ with the sphere $x^2 + y^2 + z^2 - 6x + 2y - 15 = 0$.

2. A line perpendicular to the plane of a circle through its center is called the *axis* of the circle. Find the circle: (a) which lies in the plane $z = 4$, has a radius 3 and Oz as axis; (b) which lies in the plane $y = 5$, has a radius 2 and the line $x - 3 = 0$, $z - 4 = 0$ as axis.

3. Find the circles of radius 3 on the sphere of radius 4 about the origin whose common axis is equally inclined to the coordinate axes.

4. Does the line joining the points $(2, -1, -6)$, $(-1, 2, 3)$ intersect the sphere $x^2 + y^2 + z^2 = 10$? Find the points of intersection.

5. Find the planes tangent to the following spheres at the given points: (*a*) $x^2 + y^2 + z^2 - 3y - 5z - 2 = 0$, at $(2, -1, 3)$;
(*b*) $x^2 + y^2 + z^2 + 2x - 6y + z - 1 = 0$, at $(0, 1, -3)$;
(*c*) $3(x^2 + y^2 + z^2) - 5x + 2y - z = 0$, at the origin;
(*d*) $x^2 + y^2 + z^2 - ax - by - cz = 0$, at (a, b, c).

6. Find the tangent cone: (*a*) from $(4, 1, -2)$ to $x^2 + y^2 + z^2 = 9$; (*b*) from $(2a, 0, 0)$ to $x^2 + y^2 + z^2 = a^2$; (*c*) from $(4, 4, 4)$ to $x^2 + y^2 + z^2 = 16$; (*d*) from $(1, -5, 3)$ to $x^2 + y^2 + z^2 = 9$.

7. Find the cone with vertex at the origin tangent to the sphere $(x - 2a)^2 + y^2 + z^2 = a^2$.

8. Show that, by inversion with respect to the sphere $x^2 + y^2 + z^2 = a^2$, every plane (except one through the center) is transformed into a sphere passing through the origin.

9. With respect to the sphere $x^2 + y^2 + z^2 = 25$, find the surfaces inverse to (*a*) $x = 5$, (*b*) $x - y = 0$, (*c*) $4(x^2 + y^2 + z^2) - 20x - 25 = 0$.

10. Show that by inversion with respect to the sphere $x^2 + y^2 + z^2 = a^2$ every line through the origin is transformed into itself.

11. With respect to the sphere $x^2 + y^2 + z^2 = a^2$, find the surface inverse to the plane tangent at the point $P_1(x_1, y_1, z_1)$.

12. Show that all spheres with center at the center of inversion are transformed into concentric spheres by inversion.

13. What is the curve inverse to the circle $x^2 + y^2 + z^2 = 25$, $z = 4$, with respect to the sphere $x^2 + y^2 + z^2 = 16$?

347. Poles and Polars. Let P and P' be inverse points with respect to a given sphere; then the plane π through P', at right angles to OP (O being the center of the sphere), is called the ***polar plane*** of the point P, and P is called the ***pole*** of the plane π, with respect to the sphere.

With respect to a sphere of radius a, with center at the origin,

$$x^2 + y^2 + z^2 = a^2,$$

the *equation of the polar plane of any point* $P_1(x_1, y_1, z_1)$ is readily found by observing that its distance from the origin is a^2/r_1, and that the direction cosines of its normal are equal to x_1/r_1, y_1/r_1, z_1/r_1, where $r_1^2 = x_1^2 + y_1^2 + z_1^2$; the equation is therefore

$$x_1x + y_1y + z_1z = a^2.$$

If, in particular, the point P_1 lies on the sphere, this equation, by § 343 (5), represents the tangent plane at P_1. Hence the polar plane of any point of the sphere is the tangent plane at that point; this also follows from the definition of the polar plane.

348. With respect to the same sphere the polar planes of any two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ are

$$x_1x + y_1y + z_1z = a^2 \qquad \text{and} \qquad x_2x + y_2y + z_2z = a^2.$$

Now the condition for the polar plane of P_1 to pass through P_2 is

$$x_1x_2 + y_1y_2 + z_1z_2 = a^2;$$

but this is also the condition for the polar plane of P_2 to pass through P_1 Hence *the polar planes of all the points of any plane* π (not passing through the origin O) *pass through a common point, namely, the pole of the plane* π; and conversely, *the poles of all the planes through a common point P lie in a plane, namely the polar plane of P.*

349. The polar plane of any point P of the line determined by two given points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ (always with respect to the same sphere $x^2 + y^2 + z^2 = a^2$) is

$$[x_1 + k(x_2 - x_1)]x + [y_1 + k(y_2 - y_1)]y + [z_1 + k(z_2 - z_1)]z = a^2.$$

This equation can be written in the form

$$x_1x + y_1y + z_1z - a^2 + \frac{k}{1-k}(x_2x + y_2y + z_2z - a^2) = 0,$$

which for a variable k represents the planes of the pencil whose axis is the intersection of the polar planes of P_1 and P_2. Hence *the polar planes of all the points of a line* λ *pass through a common line*; and conversely, *the poles of all the planes of a pencil lie on a line.*

Two lines related in this way are called ***conjugate lines*** (or conjugate axes, reciprocal polars). Thus the line P_1P_2

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

and the line
$$x_1x + y_1y + z_1z = a^2,$$
$$x_2x + y_2y + z_2z = a^2$$
are conjugate with respect to the sphere $x^2 + y^2 + z^2 = a^2$.

As the direction cosines of these lines are proportional to
$$x_2 - x_1, \quad y_2 - y_1, \quad z_2 - z_1$$
and
$$\begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix}, \quad \begin{vmatrix} z_1 & x_1 \\ z_2 & x_2 \end{vmatrix}, \quad \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix},$$
respectively, the two conjugate lines are at right angles (§ 331).

350. By the method used in the corresponding problem in the plane (§ 95) it can be shown that the polar plane of any point $P_1(x_1, y_1, z_1)$ with respect to any sphere
$$A(x^2 + y^2 + z^2) + 2\,Gx + 2\,Hy + 2\,Iz + J = 0$$
is
$$A(x_1x + y_1y + z_1z) + G(x_1 + x) + H(y_1 + y) + I(z_1 + z) + J = 0.$$

351. Power of a Point. If in the left-hand member of the equation of the sphere
$$(x - h)^2 + (y - j)^2 + (z - k)^2 - r^2 = 0$$
we substitute for x, y, z, the coordinates x_1, y_1, z_1 of any point *not* on the sphere, we obtain an expression $(x_1 - h)^2 + (y_1 - j)^2 + (z_1 - k)^2 - r^2$ different from zero which is called the ***power of the point*** $P_1(x_1, y_1, z_1)$ ***with respect to the sphere.***

As $(x_1 - h)^2 + (y_1 - j)^2 + (z_1 - k)^2$ is the square of the distance d between the point P_1 and the center C of the sphere, we can write the power of P_1 briefly
$$d^2 - r^2;$$
the power of P_1 is positive or negative according as P_1 lies outside or within the sphere. For a point P_1 outside, the power is evidently the square of the length of a tangent drawn from P_1 to the sphere.

352. Radical Plane, Axis, Center. The locus of a point whose powers with respect to the two spheres
$$x^2 + y^2 + z^2 + a_1x + b_1y + c_1z + d_1 = 0,$$
$$x^2 + y^2 + z^2 + a_2x + b_2y + c_2z + d_2 = 0$$
are equal is evidently the plane
$$(a_1 - a_2)x + (b_1 - b_2)y + (c_1 - c_2)z + d_1 - d_2 = 0,$$
which is called the ***radical plane*** of the two spheres. It always exists unless the two spheres are concentric.

It is easily proved that the three radical planes of any three spheres (no two of which are concentric) are planes of the same pencil (§ 323); and hence that the locus of the points of equal power with respect to three spheres is a straight line. This line is called the ***radical axis*** of the three spheres; it exists unless the centers lie in a straight line.

The six radical planes of four spheres, taken in pairs, are in general planes of a sheaf (§ 324). Hence there is in general but one point of equal power with respect to four spheres. This point, the ***radical center*** of the four spheres, exists unless the four centers lie in a plane.

353. Family of Spheres. The equation

$$(x^2+y^2+z^2+a_1x+b_1y+c_1z+d_1)+k(x^2+y^2+z^2+a_2x+b_2y+c_2z+d_2)=0$$

represents a ***family***, or ***pencil, of spheres***, provided $k \neq -1$. If the two spheres

$$x^2+y^2+z^2+a_1x+b_1y+c_1z+d_1=0,$$
$$x^2+y^2+z^2+a_2x+b_2y+c_2z+d_2=0$$

intersect, every sphere of the pencil passes through the common circle of these two spheres. If $k=-1$, the equation represents the radical plane of the two spheres.

EXERCISES

1. Find the radius of the circle in which the polar plane of the point $(4, 3, -1)$ with respect to $x^2+y^2+z^2=16$ cuts the sphere.

2. Find the radius of the circle in which the polar plane of the point $(5, -1, 2)$ with respect to $x^2+y^2+z^2-2x+4y=0$ cuts the sphere.

3. Show that the plane $3x+y-4z=19$ is tangent to the sphere $x^2+y^2+z^2-2x-4y-6z-12=0$, and find the point of contact.

4. If a point describes the plane $4x-5y-3x=16$, find the coordinates of that point about which the polar plane of the point turns with respect to the sphere $x^2+y^2+z^2=16$.

5. If a point describes the plane $2x+3y+z=4$, find that point about which the polar plane of the point turns with respect to the sphere $x^2+y^2+z^2=8$.

6. If a point describes the line $\dfrac{x-2}{3}=\dfrac{y+3}{5}=\dfrac{z-1}{-2}$, find the equations of that line about which the polar plane of the point turns with respect to the sphere $x^2+y^2+z^2=25$. Show that the two lines are perpendicular.

7. If a point describe the line $2x-3y+4z=2$, $x+y+z=3$, find the equations of that line about which the polar plane of the point turns with respect to the sphere $x^2+y^2+z^2=16$. Show that the two lines are perpendicular.

8. Find the sphere through the origin that passes through the circle of intersection of the spheres $x^2+y^2+z^2-3x+4y-5z-8=0$, $x^2+y^2+z^2-2x+y-z-10=0$.

9. Show that the locus of a point whose powers with respect to two given spheres have a constant ratio is a sphere except when the ratio is unity.

10. Show that the radical plane of two spheres is perpendicular to the line joining their centers.

11. Show that the radical plane of two spheres tangent internally or externally is their common tangent plane.

12. Find the equations of the radical axis of the spheres $x^2+y^2+z^2-3x-2y-z-4=0$, $x^2+y^2+z^2+5x-3y-2z-8=0$, $x^2+y^2+z^2-16=0$.

13. Find the radical center of the spheres $x^2+y^2+z^2-6x+2y-z+6=0$, $x^2+y^2+z^2-10=0$, $x^2+y^2+z^2+2x-3y+5z-6=0$, $x^2+y^2+z^2-2x+4y-12=0$.

14. Show that the three radical planes of three spheres are planes of the same pencil.

15. Two spheres are said to be orthogonal when their tangent planes at every point of their circle of intersection are perpendicular. Show that the two spheres $x^2+y^2+z^2+a_1x+b_1y+c_1z+d_1=0$, $x^2+y^2+z^2+a_2x+b_2y+c_2z+d_2=0$ are orthogonal when $a_1a_2+b_1b_2+c_1c_2=2(d_1+d_2)$.

16. Write the equation of the cone tangent to the sphere $x^2+y^2+z^2=r^2$ with vertex $(0, 0, z_1)$. Divide this equation by z_1^2 and let the vertex recede indefinitely, *i.e.* let z_1 increase indefinitely. The equation $x^2+y^2=r^2$, thus obtained, represents the cylinder with axis along the axis Oz and tangent to the sphere $x^2+y^2+z^2=r^2$.

17. In the equation of the tangent cone (§ 345) write for the coordinates of the vertex $x_1 = r_1 l_1$, $y_1 = r_1 m_1$, $z_1 = r_1 n_1$; divide the equation by r_1^2 and let r_1 increase indefinitely, *i.e.* let the vertex of the cone recede indefinitely. The tangent cone thus becomes a tangent cylinder with axis passing through the center of the sphere and having the direction cosines l_1, m_1, n_1. Show that this tangent cylinder is

$$(l_1 x + m_1 y + n_1 z)^2 - (x^2 + y^2 + z^2 - r^2) = 0.$$

18. From the result of Ex. 17, find the cylinder with axis equally inclined to the coordinate axes which is tangent to the sphere $x^2 + y^2 + z^2 = r^2$.

19. From the result of Ex. 17, find the cylinders with axes along the coordinate axes which are tangent to the sphere $x^2 + y^2 + z^2 = r^2$.

20. Find the cylinder with axis through the origin which is tangent to the sphere $x^2 + y^2 + z^2 - 4x + 6y - 8 = 0$.

21. Find the family of spheres inscribed in the cylinder

$$(lx + my + nz)^2 - (x^2 + y^2 + z^2 - r^2) = 0.$$

22. Find the cylinder with axis having direction cosines l, m, n which is tangent to the sphere $(x - h)^2 + (y - j)^2 + (z - k)^2 = r^2$.

23. Show that as the point P recedes indefinitely from the origin along a line through the origin of direction cosines l, m, n, the polar plane of P with respect to the sphere $x^2 + y^2 + z^2 = a^2$ becomes ultimately $lx + my + nz = 0$.

CHAPTER XVI

QUADRIC SURFACES

354. The Ellipsoid. The surface represented by the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

is called an ***ellipsoid.*** Its shape is best investigated by taking cross-sections at right angles to the axes of coordinates.

Thus the coordinate plane Oyz whose equation is $x=0$ intersects the ellipsoid in the ellipse

$$\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

Any other plane perpendicular to the axis Ox (Fig. 140), at

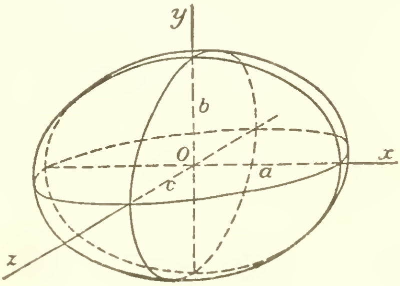


FIG. 140

the distance $h < a$ from the plane Oyz intersects the ellipsoid in an ellipse whose equation is

$$\frac{y^2}{b^2}+\frac{z^2}{c^2}=1-\frac{h^2}{a^2},$$

i.e.

$$\frac{y^2}{b^2\left(1-\frac{h^2}{a^2}\right)}+\frac{z^2}{c^2\left(1-\frac{h^2}{a^2}\right)}=1.$$

Strictly speaking this is the equation of the cylinder that projects the cross-section on the plane Oyz. But it can also be interpreted as the equation of the cross-section itself, referred to the point $(h, 0, 0)$ as origin and axes in the cross-section parallel to Oy and Oz.

Notice that as $h < a$, h^2/a^2, and hence also $1 - h^2/a^2$, is a positive proper fraction. The semi-axes $b\sqrt{1 - h^2/a^2}$, $c\sqrt{1 - h^2/a^2}$ of the cross-section are therefore less than b and c, respectively. As h increases from 0 to a, these semi-axes gradually diminish from b, c to 0.

355. Cross-Sections. Cross-sections on the opposite side of the plane Oyz give the same results: the ellipsoid is evidently symmetric with respect to the plane Oyz.

By the same method we find that cross-sections perpendicular to the axes Oy and Oz give ellipses with semi-axes diminishing as we recede from the origin. The surface is evidently symmetric to each of the coordinate planes. It follows that the origin is a ***center***, *i.e.* every chord through that point is bisected at that point. In other words, if (x, y, z) is a point of the surface, so is $(-x, -y, -z)$. Indeed, it is clear from the equation that if (x, y, z) lies on the ellipsoid, so do the seven other points $(x, y, -z)$, $(x, -y, z)$, $(-x, y, z)$, $(x, -y, -z)$, $(-x, y, -z)$, $(-x, -y, z)$, $(-x, -y, -z)$. A chord through the center is called a ***diameter***.

It follows that it suffices to study the shape of the portion of the surface contained in one octant, say that contained in the trihedral formed by the positive axes Ox, Oy, Oz; the remaining portions are then obtained by reflection in the coordinate planes.

The ellipsoid is a *closed surface*; it does not extend to infinity; indeed it is completely contained within the parallelepiped with center at the origin and edges $2a$, $2b$, $2c$, parallel to Ox, Oy, Oz, respectively.

356. Special Cases. In general, the *semi-axes* a, b, c of the ellipsoid, *i.e.* the intercepts made by it on the axes of coordinates, are different. But it may happen that two of them, or even all three, are equal.

In the latter case, *i.e.* if $a = b = c$, the ellipsoid evidently reduces to a ***sphere***.

If two of the axes are equal, *e.g.* if $b = c$, the surface

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{b^2} = 1$$

is called an ***ellipsoid of revolution*** because it can be generated by revolving the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

about the axis Ox (Fig. 141). Any cross-section at right angles to Ox, the ***axis of revolution***, is a circle, while the cross-sections at right angles to Oy and Oz are ellipses. The circular cross-section in the plane Oyz is called the ***equator;*** the intersections of the surface with the axis of revolution are the ***poles***.

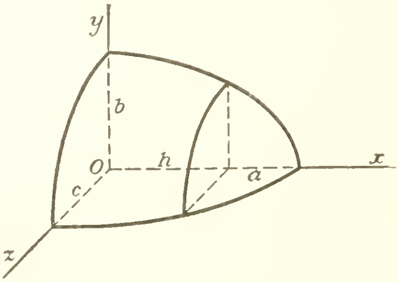


FIG. 141

If $a > b$ (a being the intercept on the axis of revolution), the ellipsoid of revolution is called ***prolate;*** if $a < b$, it is called ***oblate***. In astronomy the ellipsoid of revolution is often called ***spheroid***, the surfaces of the planets which are approximately ellipsoids of revolution being nearly spherical. Thus for the surface of the earth the major semi-axis, *i.e.* the radius of the equator, is 3962.8 miles while the minor semi-axis, *i.e.* the distance from the center to the north or south pole, is 3949.6 miles.

357. Surfaces of Revolution. A surface that can be generated by the revolution of a plane curve about a line in the plane of the curve is called a ***surface of revolution.*** Any such surface is fully determined by the generating curve and the position of the axis of revolution with respect to the curve.

Let us take the axis of revolution as axis Ox, and let the equation of the generating curve be

$$y = f(x).$$

As this curve revolves about Ox, any point P of the curve (Fig. 142) describes a circle about Ox as axis, with a radius equal to the ordinate $f(x)$ of the generating curve. For any position of P we have therefore

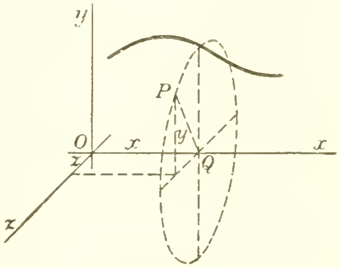


FIG. 142

$$y^2 + z^2 = [f(x)]^2,$$

and this is the *equation of the surface of revolution.*

Thus if the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

revolves about the axis Ox, we find since $y = \pm (b/a)\sqrt{a^2 - x^2}$ for the ellipsoid of revolution so generated the equation

$$y^2 + z^2 = \frac{b^2}{a^2}(a^2 - x^2),$$

which agrees with that of § 356.

Any section of a surface of revolution at right angles to the axis of revolution is of course a circle; these sections are called ***parallel circles***, or simply *parallels* (as on the earth's surface). Any section of a surface of revolution by a plane passing through the axis of revolution is called a ***meridian section;*** it consists of the generating curve and its reflection in the axis of revolution.

EXERCISES

1. An ellipsoid has six *foci*, viz. the foci of the three ellipses in which the ellipsoid is intersected by its planes of symmetry. Determine the coordinates of these foci: (*a*) for an ellipsoid with semi-axes 1, 2, 3; (*b*) for the earth (see § 356); (*c*) for an ellipsoid of semi-axes 10, 8, 1; (*d*) for an ellipsoid of semi-axes 1, 1, 5.

2. Show that the intersection of an ellipsoid with any plane actually cutting the ellipsoid is an ellipse by proving that the projection of this curve of intersection on each coordinate plane is an ellipse.

3. Assuming $a > b > c$ in the equation of § 354 find the planes through Oy that intersect the ellipsoid in circles.

4. Find the equation of the paraboloid of revolution generated by the revolution of the parabola $y^2 = 4\,ax$ about Ox.

5. Find the equation of a *torus*, or *anchor-ring*, *i.e.* the surface generated by the revolution of a circle of radius a about a line in its plane at the distance $b > a$ from its center.

6. Find the equation of the surface generated by the revolution of a circle of radius a about a line in its plane at the distance $b < a$ from its center. Is the appearance of this surface noticeably different from the surface of Ex. 5?

7. Show what happens to the surface of Ex. 6 when $b = 0$; when $b = a$.

8. Find the equation of the surface generated by the revolution of the parabola $y^2 = 4\,ax$ about: (*a*) the tangent at the vertex; (*b*) the latus rectum.

9. Find the equation of the surface generated by the revolution of the hyperbola $xy = a^2$ about an asymptote.

10. Find the cone generated by the revolution of the line $y = mx + b$ about: (*a*) Ox. (*b*) Oy.

11. How are the following surfaces of revolution generated?
(*a*) $y^2+z^2=x^4$. (*b*) $2\,x^2+2\,y^2-3\,z=0$. (*c*) $x^2+y^2-z^2-2\,x+4=0$.

12. Find the equation of the surface generated by the revolution of the ellipse $x^2 + 4\,y^2 - 4\,x = 0$: (*a*) about the major axis; (*b*) about the minor axis; (*c*) about the tangent at the origin.

358. Hyperboloid of One Sheet. The surface represented by the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

is called a ***hyperboloid of one sheet*** (Fig. 143). The intercepts on the axes Ox, Oy are $\pm a$, $\pm b$; the axis Oz does not intersect the surface.

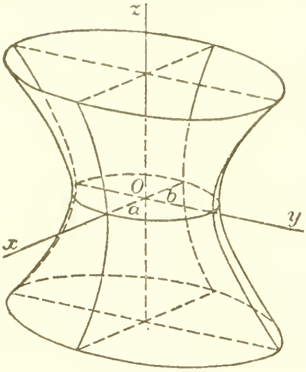


FIG. 143

359. Cross-Sections. The plane Oxy intersects the surface in the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1;$$

cross-sections perpendicular to Oz give ellipses with ever-increasing semi-axes.

The planes Oyz and Ozx intersect the surface in the hyperbolas

$$\frac{y^2}{b^2}-\frac{z^2}{c^2}=1,\quad \frac{x^2}{a^2}-\frac{z^2}{c^2}=1.$$

Any plane perpendicular to Ox, at the distance h from the origin, intersects the hyperboloid in a hyperbola, viz.

$$\frac{y^2}{b^2\left(1-\frac{h^2}{a^2}\right)}-\frac{z^2}{c^2\left(1-\frac{h^2}{a^2}\right)}=1.$$

z

As long as $h < a$ this hyperbola has its transverse axis parallel to Oy while for $h > a$ the transverse axis is parallel to Oz; for $h = a$ the equation reduces to $y^2/b^2 - z^2/c^2 = 0$ and represents two straight lines, viz. the parallels through $(a, 0, 0)$ to the asymptotes of the hyperbola $y^2/b^2 - z^2/c^2 = 1$ which is the intersection of the surface with the plane Oyz.

Similar considerations apply to the cross-sections perpendicular to Oy.

The hyperboloid has the same properties of symmetry as the ellipsoid (§ 355); the origin is a *center*, and it suffices to investigate the shape of the surface in one octant.

360. Hyperboloid of Revolution of one Sheet. If in the hyperboloid of one sheet we have $a = b$, the cross-sections perpendicular to the axis Oz are all circles so that the surface can be generated by the revolution of the hyperbola

$$\frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

about Oz. Such a surface is called a ***hyperboloid of revolution of one sheet.***

361. Other Forms. The equations

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \quad -\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

also represent hyperboloids of one sheet which can be investigated as in §§ 358–360. In the former of these the axis Oy, in the latter the axis Ox, does not meet the surface.

Every hyperboloid of one sheet extends to infinity.

362. Hyperboloid of Two Sheets. The surface represented by the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

is called a ***hyperboloid of two sheets*** (Fig. 144).

The intercepts on Ox are $\pm a$; the axes Oy, Oz do not meet the surface.

363. Cross-Sections. The cross-sections at right angles to Ox, at the distance h from the origin are

$$-\frac{y^2}{b^2\left(1-\frac{h^2}{a^2}\right)}-\frac{z^2}{c^2\left(1-\frac{h^2}{a^2}\right)}=1;$$

these are imaginary as long as $h < a$; for $h > a$ they are ellipses with ever-increasing semi-axes as we recede from the origin.

The cross-sections at right angles to Oy and Oz are hyperbolas.

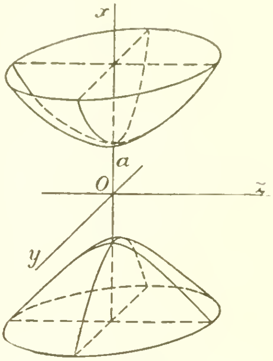


FIG. 144

The hyperboloid of two sheets, like that of one sheet and like the ellipsoid, has three mutually rectangular planes of symmetry whose intersection is therefore a *center*.

The surfaces

$$-\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1,\quad -\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

are hyperboloids of two sheets, the former being met by Oy, the latter by Oz, in real points.

The hyperboloid of two sheets extends to infinity.

364. Hyperboloid of Revolution of Two Sheets. If $b = c$ in the equation of § 362, the cross-sections at right angles to Ox are circles and the surface becomes a *hyperboloid of revolution of two sheets*.

365. Imaginary Ellipsoid. The equation

$$-\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

is not satisfied by any point with real coordinates. It is sometimes said to represent an *imaginary ellipsoid*.

366. The Paraboloids. The surfaces

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=2cz, \quad \frac{x^2}{a^2}-\frac{y^2}{b^2}=2cz,$$

which are called the *elliptic paraboloid* (Fig. 145) and *hyperbolic paraboloid* (Fig. 146), respectively, have each only two planes of symmetry, viz. the planes Oyz and Ozx. We here assume that $c \neq 0$. The cross-sections at right angles to the axis Oz are evidently ellipses in the case of the elliptic paraboloid, and hyperbolas in the case of the hyperbolic paraboloid. The plane Oxy itself has only the origin in common with the elliptic paraboloid; it intersects the hyperbolic paraboloid in the two lines $z = 0$, $x^2/a^2 - y^2/b^2 = 0$, *i.e.*, $y = \pm bx/a$.

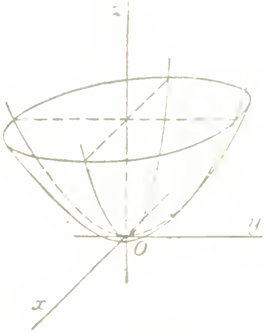


Fig. 145

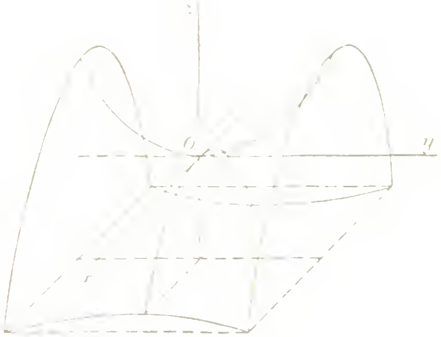


Fig. 146

The intersections of the elliptic paraboloid (Fig. 145) with the planes Oyz and Ozx are parabolas with Oz as axis and O as vertex, opening in the sense of positive z if c is positive, in the sense of negative z if c is negative. Planes parallel to these coordinate planes intersect the elliptic paraboloid in parabolas with axes parallel to Oz, but with vertices not on the axes Ox, Oy, respectively.

For the hyperbolic paraboloid (Fig. 146), which is saddle-shaped at the origin, the intersections with the planes Oyz and

Ozx are also parabolas with Oz as axis: if c is positive the parabola in the plane Oyz opens in the sense of negative z, that in the plane Ozx opens in the sense of positive z. Similarly for the parallel sections.

367. Paraboloid of Revolution. If in the equation of the elliptic paraboloid we have $a = b$, it reduces to the form

$$x^2 + y^2 = 2\,pz.$$

This represents a surface of revolution, called the ***paraboloid of revolution***. This surface can be regarded as generated by the revolution of the parabola $y^2 = 2\,pz$ about the axis Oz.

368. Elliptic Cone. The surface represented by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$$

is an ***elliptic cone***, with the origin as vertex and the axis Oz as axis (Fig. 147).

The plane Oxy has only the origin in common with the surface. Every parallel plane $z = k$, whether k be positive or negative, intersects the surface in an ellipse, with semi-axes increasing proportionally to k.

The plane Oyz, as well as the plane Ozx, intersects the surface in two straight lines through the origin. Every plane parallel to Oyz or to Ozx intersects the surface in a hyperbola.

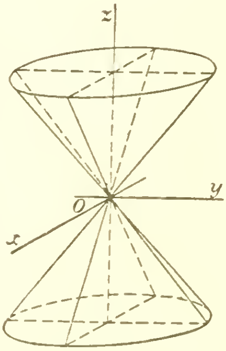


FIG. 147

369. Circular Cone. If in the equation of the elliptic cone we have $a = b$, the cross-sections at right angles to the axis Oz become circles. The cone is then an ordinary ***circular cone***, or

cone of revolution, which can be generated by the revolution of the line $y = (a/c)z$ about the axis Oz. Putting $a/c = m$ we can write the equation of a cone of revolution about Oz, with vertex at O, in the form

$$x^2 + y^2 = m^2z^2.$$

370. Quadric Surfaces. The ellipsoid, the two hyperboloids, the two paraboloids, and the elliptic cone are called ***quadric surfaces*** because their cartesian equations are all of the second degree.

Let us now try to determine, conversely, all the various loci that can be represented by the ***general equation of the second degree***

$$Ax^2 + By^2 + Cz^2 + 2\,Dyz + 2\,Ezx + 2\,Fxy \\ + 2\,Gx + 2\,Hy + 2\,Iz + J = 0.$$

In studying the equation of the second degree in x and y (§ 249) it was shown that the term in xy can always be removed by turning the axes about the origin through a certain angle. Similarly, it can be shown in the case of three variables that by a properly selected rotation of the coordinate trihedral about the origin the terms in yx, zx, xy can in general all be removed so that the equation reduces to the form

$$(1) \qquad \boldsymbol{Ax^2 + By^2 + Cz^2 + 2\,Gx + 2\,Hy + 2\,Iz + J = 0.}$$

This transformation being somewhat long will not be given here. We shall proceed to classify the surfaces represented by equations of the form (1).

371. Classification. The equation (1) can be further simplified by completing the squares. *Three cases* may be distinguished according as the coefficients A, B, C are all three different from zero, one only is zero, or two are zero.

CASE (a): $A \neq 0$, $B \neq 0$, $C \neq 0$. Completing the squares in x, y, z we find

$$A\left(x+\frac{G}{A}\right)^2+B\left(y+\frac{H}{B}\right)^2+C\left(z+\frac{I}{C}\right)^2=\frac{G^2}{A}+\frac{H^2}{B}+\frac{I^2}{C}-J=J_1.$$

Referred to parallel axes through the point $(-G/A, -H/B, -I/C)$ this equation becomes

$$(2) \qquad Ax^2+By^2+Cz^2=J_1.$$

CASE (b): $A \neq 0$, $B \neq 0$, $C=0$. Completing the squares in x and y we find

$$A\left(x+\frac{G}{A}\right)^2+B\left(y+\frac{H}{B}\right)^2+2\,Iz=\frac{G^2}{A}+\frac{H^2}{B}-J=J_2.$$

If $I \neq 0$ we can transform to parallel axes through the point $(-G/A, -H/B, J_2/2\,I)$ so that the equation becomes

$$(3) \qquad Ax^2+By^2+2\,Iz=0.$$

If, however, $I=0$, we obtain by transforming to the point $(-G/A, -H/B, 0)$

$$(3') \qquad Ax^2+By^2=J_2.$$

CASE (c): $A \neq 0$, $B=0$, $C=0$. Completing the square in x we have

$$A\left(x+\frac{G}{A}\right)^2+2\,Hy+2\,Iz=\frac{G^2}{A}-J=J_3.$$

If H and I are not both zero we can transform to parallel axes through the point $(-G/A, J_3/2\,H, 0)$ or through $(-G/A, 0, J_3/2\,I)$ and find

$$(4) \qquad Ax^2+2\,Hy+2\,Iz=0.$$

If $H=0$ and $I=0$ we transform to the point $(-G/A, 0, 0)$ so that we find

$$(4') \qquad Ax^2=J_3.$$

372. Squared Terms all Present. Case (a). We proceed to discuss the loci represented by (2). If $J_1 \neq 0$, we can divide (2) by J_1 and obtain:

(α) if A/J_1, B/J_1, C/J_1 are positive, an *ellipsoid* (§ 354);

(β) if two of these coefficients are positive while the third is negative, a *hyperboloid of one sheet* (§ 358);

(γ) if one coefficient is positive while two are negative, a *hyperboloid of two sheets* (§ 362);

(δ) if all three coefficients are negative, the equation is not satisfied by any real point (§ 365);

If $J_1 = 0$ the equation (2) represents an *elliptic cone* (§ 368) unless A, B, C all have the same sign, in which case *the origin* is the only point represented.

373. Case (b). The equation (3) of § 371 evidently furnishes the two *paraboloids* (§ 366): the paraboloid is *elliptic* if A and B have the same sign; it is *hyperbolic* if A and B are of opposite sign.

The equation (3′) since it does not contain z and hence leaves z arbitrary represents the *cylinder, with generators parallel to Oz, passing through the conic* $Ax^2 + By^2 = J_2$. As A and B are assumed different from zero, this conic is an ellipse if A/J_2 and and B/J_2 are both positive, a hyperbola if A/J_2 and B/J_2 are of opposite sign, and it is imaginary if A/J_2 and B/J_2 are both negative. This assumes $J_2 \neq 0$. If $J_2 = 0$, the conic degenerates into two straight lines, real or imaginary; the cylinder degenerates into two planes if the lines are real.

374. Case (c). There remain equations (4) and (4′). To simplify (4) we may turn the coordinate trihedral about Ox through an angle whose tangent is $-H/I$; this is done by putting

$$x = x', \quad y = \frac{Iy' + Hz'}{\sqrt{H^2 + I^2}}, \quad z = \frac{-Hy' + Iz'}{\sqrt{H^2 + I^2}};$$

our equation then becomes

$$Ax'^2 + 2\sqrt{H^2 + I^2}\,z' = 0.$$

It evidently represents a *parabolic cylinder*, with generators parallel to Oy.

Finally, the equation (4′) is readily seen to represent *two planes* perpendicular to Ox, real or imaginary, unless $J_3 = 0$ in which case it represents the plane Oyz.

EXERCISES

1. Name and locate the following surfaces:

(*a*) $x^2 + 2y^2 + 3z^2 = 4.$ (*b*) $x^2 + y^2 - 5z - 6 = 0.$

(*c*) $x^2 - y^2 + z^2 = 4.$ (*d*) $x^2 - y^2 + z^2 + 3z + 6 = 0.$

(*e*) $2y^2 - 4z^2 - 5 = 0.$ (*f*) $2x^2 + y^2 + 3z^2 + 5 = 0.$

(*g*) $5z^2 + 2x^2 = 10.$ (*h*) $z^2 - 9 = 0.$

(*i*) $x^2 - y + 1 = 0.$ (*j*) $x^2 - y^2 - z^2 + 6z = 9.$

(*k*) $x^2 + 3y^2 + z^2 + 4z + 4 = 0.$ (*l*) $z^2 + y - 9 = 0.$

2. The cone

$$x^2/a^2 + y^2/b^2 - z^2/c^2 = 0$$

is called the *asymptotic cone* of the hyperboloid of one sheet

$$x^2/a^2 + y^2/b^2 - z^2/c^2 = 1.$$

Show that as z increases the two surfaces approach each other, *i.e.* they bear a relation similar to a hyperbola and its asymptotes.

3. What is the asymptotic cone of the hyperboloid of two sheets?

4. Show that the intersection of a hyperboloid of two sheets with any plane actually cutting the surface is an ellipse, parabola, or hyperbola. Determine the position of the plane for each conic.

5. Show that in general nine points determine a quadric surface and that the equation may be written as a determinant of the tenth order equated to zero.

6. Show that the surface inverse to the cylinder $x^2 + y^2 = a^2$, with respect to the sphere $x^2 + y^2 + z^2 = a^2$, is the torus generated by the revolution of the circle $(y - a/2)^2 + z^2 = a^2$ about the axis Ox.

7. Determine the nature of the surface $xyz = a^3$ by means of cross-sections.

375. Tangent Plane to the Ellipsoid. The plane tangent to the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

can be found as follows (compare §§ 344, 345). The equations of the line joining any two given points (x_1, y_1, z_1) and (x_2, y_2, z_2) are

$$x=x_1+k(x_2-x_1),\quad y=y_1+k(y_2-y_1),\quad z=z_1+k(z_2-z_1).$$

This line will be tangent to the ellipsoid if the quadratic in k

$$\frac{[x_1+k(x_2-x_1)]^2}{a^2}+\frac{[y_1+k(y_2-y_1)]^2}{b^2}+\frac{[z_1+k(z_2-z_1)]^2}{c^2}=1$$

has equal roots. Writing this quadratic in the form

$$\left[\frac{(x_2-x_1)^2}{a^2}+\frac{(y_2-y_1)^2}{b^2}+\frac{(z_2-z_1)^2}{c^2}\right]k^2$$
$$+2\left[\frac{x_1(x_2-x_1)}{a^2}+\frac{y_1(y_2-y_1)}{b^2}+\frac{z_1(z_2-z_1)}{c^2}\right]k+\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}-1\right)=0,$$

we find the condition

$$\left[\left(\frac{x_1x_2}{a^2}+\frac{y_1y_2}{b^2}+\frac{z_1z_2}{c^2}-1\right)-\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}-1\right)\right]^2$$
$$=\left[\frac{(x_2-x_1)^2}{a^2}+\frac{(y_2-y_1)^2}{b^2}+\frac{(z_2-z_1)^2}{c^2}\right]\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}-1\right).$$

If now we keep the point (x_1, y_1, z_1) fixed, but let the point (x_2, y_2, z_2) vary subject to this condition, it will describe the cone, with vertex (x_1, y_1, z_1), tangent to the ellipsoid; to indicate this we shall drop the subscripts of x_2, y_2, z_2. If, in particular, the point (x_1, y_1, z_1) be chosen on the ellipsoid, we have

$$\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}=1,$$

and the cone becomes the tangent plane. The *equation of the tangent plane to the ellipsoid at the point* (x_1, y_1, z_1) is, therefore:

$$\frac{x_1x}{a^2}+\frac{y_1y}{b^2}+\frac{z_1z}{c^2}=1.$$

376. Tangent Planes to Hyperboloids. In the same way it can be shown that the *tangent planes to the hyperboloids*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1, \quad \frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

at (x_1, y_1, z_1) are

$$\frac{x_1x}{a^2}+\frac{y_1y}{b^2}-\frac{z_1z}{c^2}=1, \quad \frac{x_1x}{a^2}-\frac{y_1y}{b^2}-\frac{z_1z}{c^2}=1.$$

By an equally elementary, but somewhat longer, calculation it can be shown that the *tangent plane to the quadric surface*

$$Ax^2+By^2+Cz^2+2\,Dyz+2\,Ezx+2\,Fxy \\ +2\,Gx+2\,Hy+2\,Iz+J=0$$

at (x_1, y_1, z_1) is:

$$Ax_1x+By_1y+Cz_1z+D(y_1z+z_1y)+E(z_1x+x_1z)+F(x_1y+y_1x) \\ +G(x_1+x)+H(y_1+y)+I(z_1+z)+J=0.$$

In particular, the *tangent planes to the paraboloids*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=2\,cz, \quad \frac{x^2}{a^2}-\frac{y^2}{b^2}=2\,cz$$

are

$$\frac{x_1x}{a^2}+\frac{y_1y}{b^2}=c(z_1+z), \quad \frac{x_1x}{a^2}-\frac{y_1y}{b^2}=c(z_1+z).$$

377. Ruled Surfaces. A surface that can be generated by the motion of a straight line is called a *ruled surface;* the line is called the *generator.*

The plane is a ruled surface. Among the quadric surfaces not only the cylinders and cones but also the hyperboloid of one sheet and the hyperbolic paraboloid are ruled surfaces.

378. Rulings on a Hyperboloid of One Sheet. To show this for the hyperboloid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1,$$

we write the equation in the form

$$\frac{y^2}{b^2}-\frac{z^2}{c^2}=1-\frac{x^2}{a^2}$$

and factor both members:

$$\left(\frac{y}{b}+\frac{z}{c}\right)\left(\frac{y}{b}-\frac{z}{c}\right)=\left(1+\frac{x}{a}\right)\left(1-\frac{x}{a}\right).$$

It is then apparent that any point whose coordinates satisfy the two equations

$$\frac{y}{b}+\frac{z}{c}=k\left(1+\frac{x}{a}\right), \quad \frac{y}{b}-\frac{z}{a}=\frac{1}{k}\left(1-\frac{x}{a}\right),$$

where k is an arbitrary parameter, lies on the hyperboloid. These two equations represent for every value of k $(\neq 0)$ a straight line. The hyperboloid of one sheet contains therefore the family of lines represented by the last two equations with variable k.

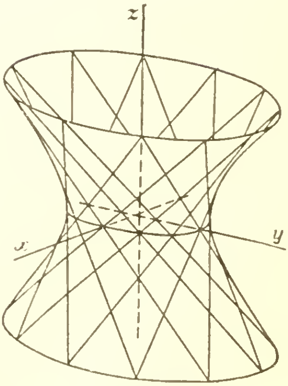


FIG. 148

In exactly the same way it is shown that the same hyperboloid also contains the family of lines

$$\frac{y}{b}-\frac{z}{c}=k'\left(1+\frac{x}{a}\right), \quad \frac{y}{b}+\frac{z}{c}=\frac{1}{k'}\left(1-\frac{x}{a}\right).$$

Thus *every hyperboloid of one sheet contains two sets of rectilinear generators* (Fig. 148).

379. Rulings on a Hyperbolic Paraboloid. The hyperbolic paraboloid (Fig. 149)

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=2\,cz$$

also *contains two sets of rectilinear generators*, namely,

$$\frac{x}{a}+\frac{y}{b}=k\cdot 2\,cz,\quad \frac{x}{a}-\frac{y}{b}=\frac{1}{k},$$

and

$$\frac{x}{a}-\frac{y}{b}=k'\cdot 2\,cz,\quad \frac{x}{a}+\frac{y}{b}=\frac{1}{k'}.$$

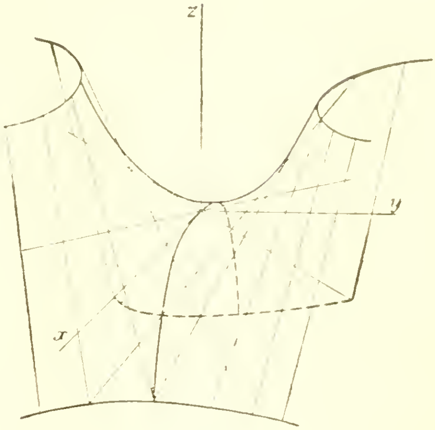


FIG. 149

EXERCISES

1. Derive the equation of the tangent plane to:

(*a*) the elliptic paraboloid; (*b*) the hyperbolic paraboloid; (*c*) the elliptic cone.

2. The line perpendicular to a tangent plane at a point of contact is called the *normal line*. Write the equations of the tangent planes and normal lines to the following quadric surfaces at the points indicated:

(*a*) $x^2/9+y^2/4-z^2/16=1$, at $(3,\,-1,\,2)$;

(*b*) $x^2+2\,y^2+z^2=10$, at $(2,\,1,\,-2)$;

(*c*) $x^2+2\,y^2-2\,z^2=0$, at $(4,\,1,\,3)$; (*d*) $x^2-3\,y^2-z=0$, at the origin.

3. Show that the cylinder whose axis has the direction cosines l, m, n and which is tangent to the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$, is

$$\left(\frac{lx}{a^2}+\frac{my}{b^2}+\frac{nz}{c^2}\right)^2-\left(\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2}\right)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)=0.$$

4. Show that the plane $lx+my+nz=\sqrt{l^2a^2+m^2b^2+n^2c^2}$ is tangent to the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$.

5. Show that the locus of the intersection of three mutually perpendicular tangent planes to the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$, is the sphere (called *director sphere*) $x^2+y^2+z^2=a^2+b^2+c^2$.

6. Show that the elliptic cone is a ruled surface.

7. Show that any two linear equations which contain a parameter represent the generating line of a ruled surface. What surfaces are generated by the following lines?

$$(a)\ x - y + kz = 0,\ x + y - z/k = 0;\ (b)\ 3x - 4y = k,\ (3x+4y)k = 1;$$
$$(c)\ x - y + 3kz = 3k,\ k(x + y) - z = 3.$$

8. Show that every generating line of the hyperbolic paraboloid $x^2/a^2 - y^2b^2 = 2cz$ is parallel to one of the planes $x^2/a^2 - y^2/b^2 = 0$.

380. Surfaces in General. When it is required to determine the shape of a surface from its cartesian equation

$$F(x, y, z) = 0,$$

the most effective methods, apart from the calculus, are the transformation of coordinates and the taking of cross-sections, generally (though not necessarily always) at right angles to the axes of coordinates. Both these methods have been applied repeatedly to the quadric surfaces in the preceding articles.

381. Cross-Sections. The method of cross-sections is extensively used in the applications. The railroad engineer determines thus the shape of a railroad dam; the naval architect uses it in laying out his ship; even the biologist uses it in constructing enlarged models of small organs of plants or animals.

382. Parallel Planes. When the given equation contains only one of the variables x, y, z, it represents of course a set of *parallel planes* (real or imaginary), at right angles to one of the axes. Thus any equation of the form

$$F(x) = 0$$

represents planes at right angles to Ox, of which as many are real as the equation has real roots.

383. Cylinders. When the given equation contains only two variables it represents a *cylinder* at right angles to one of the coordinate planes. Thus any equation of the form

$$F(x, y) = 0$$

represents a cylinder passing through the curve $F(x, y) = 0$ in the plane Oxy, with generators parallel to Oz. If, in particular, $F(x, y)$ is homogeneous in x and y, *i.e.* if all terms are of the same degree, the cylinder breaks up into planes.

384. Cones. When the given equation $F(x, y, z) = 0$ is homogeneous in x, y, and z, *i.e.* if all terms are of the same degree, the equation represents a general *cone*, with vertex at the origin. For in this case, if (x, y, z) is a point of the surface, so is the point (kx, ky, kz), where k is any constant; in other words, if P is a point of the surface, then every point of the line OP belongs to the surface; the surface can therefore be generated by the motion of a line passing through the origin.

385. Functions of Two Variables. Just as plane curves are used to represent functions of a single variable, so surfaces can be used to represent ***functions of two variables***. Thus to obtain an intuitive picture of a given function $f(x, y)$ we may construct a *model* of the surface

$$z = f(x, y),$$

such as the relief map of a mountainous country. The ordinate z of the surface represents the function.

386. Contour Lines. To obtain some idea of such a surface by means of a *plane* drawing the method of ***contour lines*** or ***level lines*** can be used. This is done, *e.g.*, in topographical maps. The method consists in taking horizontal cross-sections at equal intervals and projecting these cross-sections on the horizontal plane. Where the level lines crowd together the surface is steep; where they are relatively far apart the surface is flat.

EXERCISES

1. What surfaces are represented by the following equations?

(*a*) $Ax + By + C = 0$. (*b*) $x \cos \beta + y \sin \beta = p$.
(*c*) $y^2 + z^2 = a^2$. (*d*) $z^2 - x^2 = a^2$.
(*e*) $zx = a^2$. (*f*) $z^2 = 4\,ay$.
(*g*) $x^3 - 3\,x^2 - x + 3 = 0$. (*h*) $xyz = 0$.
(*i*) $y = x^2 - x - 6$. (*j*) $yz^2 - 9\,y = 0$.
(*k*) $x^2 + 2\,y^2 = 0$. (*l*) $x^2 = yz$.
(*m*) $x^2 - y^2 = z^2$. (*n*) $y^2 + 2\,z^2 + 4\,zx = 0$.
(*o*) $(x-1)(y-2)(z-3) = 0$. (*p*) $x^3 + y^3 - 3\,xyz = 0$.

2. Determine the nature of the following surfaces by sketching the contour lines:

(*a*) $z = x + y$. (*b*) $z = xy$. (*c*) $z = y/x$. (*d*) $z = x^2 + y^2$.
(*e*) $z = x^2 - y^2 + 4$. (*f*) $z = x^2$. (*g*) $z = x^2 + y^2 - 4\,x$. (*h*) $z = xy - x$.
(*i*) $z = 2^x$. (*j*) $y = z^2 - 4\,x$. (*k*) $y = 3\,z^2 + x^2$. (*l*) $z = 3\,x + y^2$.

3. The Cassinian ovals (§ 270) are contour lines of what surface?

4. What can be said about the nature of the contour lines of a surface $z = f(x)$? Discuss in particular: (*a*) $z = x^2 - 9$; (*b*) $z = x^3 - 8$; (*c*) $y = z^2 + 2\,z$.

387. Rotation of Coordinate Trihedral. To transform the equation of a surface from one coordinate trihedral $Oxyz$ to another $Ox'y'z'$, with the same origin O, we must find expressions for the *old* coordinates x, y, z of any point P in terms of the *new* coordinates x', y', z'. We here confine ourselves to the case when each trihedral is trirectangular; this is the case of *orthogonal transformation*, or *orthogonal substitution*.

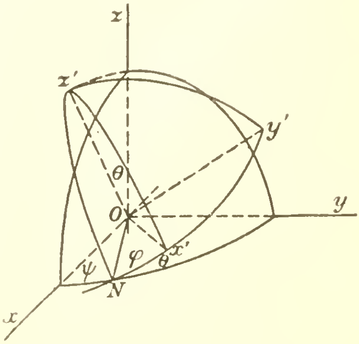


FIG. 150

Let l_1, m_1, n_1, be the direction cosines of the new axis Ox' with respect to the old axes Ox, Oy, Oz (Fig. 150); similarly l_2, m_2, n_2 those of Oy', and l_3, m_3, n_3 those of Oz'. This is indicated by the scheme

$$\begin{array}{c|ccc} & x' & y' & z' \\ \hline x & l_1 & l_2 & l_3 \\ y & m_1 & m_2 & m_3 \\ z & n_1 & n_2 & n_3 \end{array},$$

which shows at the same time that then the direction cosines of the old axis Ox with respect to the new axes Ox', Oy', Oz' are l_1, l_2, l_3, etc.

388. The nine direction cosines $l_1, l_2, \cdots n_3$ are sufficient to determine the position of the new trihedral $Ox'y'z'$ with respect to the old. But these nine quantities cannot be selected arbitrarily; they are connected by six independent relations which can be written in either of the equivalent forms

$$(1) \qquad \begin{aligned} l_1^2 + m_1^2 + n_1^2 &= 1, & l_2l_3 + m_2m_3 + n_2n_3 &= 0, \\ l_2^2 + m_2^2 + n_2^2 &= 1, & l_3l_1 + m_3m_1 + n_3n_1 &= 0, \\ l_3^2 + m_3^2 + n_3^2 &= 1, & l_1l_2 + m_1m_2 + n_1n_2 &= 0, \end{aligned}$$

or

$$(1') \qquad \begin{aligned} l_1^2 + l_2^2 + l_3^2 &= 1, & m_1n_1 + m_2n_2 + m_3n_3 &= 0, \\ m_1^2 + m_2^2 + m_3^2 &= 1, & n_1l_1 + n_2l_2 + n_3l_3 &= 0, \\ n_1^2 + n_2^2 + n_3^2 &= 1, & l_1m_1 + l_2m_2 + l_3m_3 &= 0. \end{aligned}$$

The meaning of these equations follows from §§ 297 and 300. Thus the first of the equations (1) expresses the fact that l_1, m_1, n_1 are the direction cosines of a line, viz. Ox'; the last of the equations (1′) expresses the perpendicularity of the axes Ox and Oy; and so on.

389. If x, y, z are the old, x', y', z' the new coordinates of one and the same point, we find by observing that the projection on Ox of the radius vector of P is equal to the sum of the projections on Ox of its components x', y', z' (§ 294), and similarly for the projections on Oy and Oz:

$$(2) \qquad \begin{aligned} x &= l_1x' + l_2y' + l_3z', \\ y &= m_1x' + m_2y' + m_3z', \\ z &= n_1x' + n_2y' + n_3z'. \end{aligned}$$

Indeed, these relations can be directly read off from the scheme of direction cosines in § 387.

Likewise, projecting on Ox', Oy', Oz', we find

$$(2') \qquad \begin{aligned} x' &= l_1x + m_1y + n_1z, \\ y' &= l_2x + m_2y + n_2z, \\ z' &= l_3x + m_3y + n_3z. \end{aligned}$$

As the equations (2), by means of which we can transform the equation of any surface from one rectangular system of coordinates to any other with the same origin, give x, y, z as *linear* functions of x', y', z', it follows that *such a transformation cannot change the degree of the equation of the surface.*

390. The equation (2′) must of course result also by solving the equations (2) for x', y', z', and *vice versa*. Putting

$$\begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix} = D,$$

solving (2) for x', y', z', and comparing the coefficients of x, y, z with those in (2′) we find the following relations:

$$Dl_1 = m_2n_3 - m_3n_2, \quad Dm_1 = n_2l_3 - n_3l_2, \quad Dn_1 = l_2m_3 - l_3m_2, \quad \text{etc.}$$

Squaring and adding the first three equations (compare Ex. 3, p. 45) and applying the relations (1) we find: $D^2 = 1$.

By § 321, D can be interpreted as six times the volume of the tetrahedron whose vertices are the origin and the points x', y', z' in Fig. 150, *i.e.* the intersections of the new axes with the unit sphere about the origin. The determinant gives this volume with the sign + or − according as the trihedral $Ox'y'z'$ is superposable or not (in direction and sense) to the trihedral $Oxyz$ (see § 391). It follows that $D = \pm 1$ and

$$\begin{aligned} l_1 &= \pm (m_2n_3 - m_3n_2), & m_1 &= \pm (n_2l_3 - n_3l_2), & n_1 &= \pm (l_2m_3 - l_3m_2), \\ l_2 &= \pm (m_3n_1 - m_1n_3), & m_2 &= \pm (n_3l_1 - n_1l_3), & n_2 &= \pm (l_3m_1 - l_1m_3), \\ l_3 &= \pm (m_1n_2 - m_2n_1), & m_3 &= \pm (n_1l_2 - n_2l_1), & n_3 &= \pm (l_1m_2 - l_2m_1), \end{aligned}$$

the upper or lower signs to be used according as the trihedrals are superposable or not.

391. A rectangular trihedral $Oxyz$ is called ***right-handed*** if the rotation that turns Oy through 90° into Oz appears *counterclockwise* as seen from Ox; otherwise it is called ***left-handed***. In the present work right-handed sets of axes have been used throughout.

Two right-handed as well as two left-handed rectangular trihedrals are superposable; a right-handed and a left-handed trihedral are not superposable. The difference is of the same kind as that between the gloves of the right and left hand.

Two non-superposable rectangular trihedrals become superposable upon reversing one (or all three) of the axes of either one.

392. The fact that the nine direction cosines are connected by six relations (§ 388) suggests that it must be possible to determine the position of the new trihedral with respect to the old by only three angles. As such we may take, in the case of superposable trihedrals, the angles θ, ϕ, ψ, marked in Fig. 150, which are known as ***Euler's angles.***

The figure shows the intersections of the two trihedrals with a sphere of radius 1 described about the origin as center. If ON is the intersection of the planes Oxy and $Ox'y'$, Euler's angles are defined as

$$\theta = zOz', \quad \phi = NOx', \quad \psi = xON.$$

The line ON is called the *line of nodes*, or the ***nodal line.***

Imagine the new trihedral $Ox'y'z'$ initially coincident with the old trihedral $Oxyz$, in direction and sense. Now turn the new trihedral about Oz in the positive (counterclockwise) sense until Ox' coincides with the assumed positive sense of the nodal line ON; the amount of this rotation gives the angle ψ. Next turn the new trihedral about ON in the positive sense until the plane $Ox'y'$ assumes its final position; this gives the angle θ as the angle between the planes Oxy and $Ox'y'$, or the angle zOz' between their normals. Finally a rotation of the new trihedral about the axis Oz', which has reached its final position, in the positive sense until Ox' assumes its final position, determines the angle ϕ.

393. The relations between the nine direction cosines and the three angles of Euler are readily found from Fig. 150 by applying the fundamental formula of spherical trigonometry $\cos c = \cos a \cos b + \sin a \sin b \cos \gamma$ successively to the spherical triangles

$$\begin{matrix} xNx', & xNy', & xNz', \\ yNx', & yNy', & yNz', \\ zNx', & zNy', & zNz'. \end{matrix}$$

We find in this way:

$$\begin{aligned} l_1 &= \cos\psi\cos\phi - \sin\psi\sin\phi\cos\theta, \\ m_1 &= \sin\psi\cos\phi + \cos\psi\sin\phi\cos\theta, \\ n_1 &= \sin\phi\sin\theta, \end{aligned}$$

$$\begin{aligned} l_2 &= -\cos\psi\sin\phi - \sin\psi\cos\phi\cos\theta, & l_3 &= \sin\psi\sin\theta, \\ m_2 &= -\sin\psi\sin\phi + \cos\psi\cos\phi\cos\theta, & m_3 &= -\cos\psi\sin\theta, \\ n_2 &= \cos\phi\sin\theta, & n_3 &= \cos\theta. \end{aligned}$$

APPENDIX

NOTE ON ABRIDGED NUMERICAL MULTIPLICATION AND DIVISION

1. In *multiplying* two numbers it is convenient to write the multiplier not below but to the right of the multiplicand in the same line with it, and to begin the formation of the partial products with the highest figure (and not with the lowest). The most important part of the product is thus obtained first. The partial products must then be moved out toward the right (and not to the left). Thus:

$$\begin{array}{r|l} \underline{35702} & \underline{87025} \\ 285616 & \\ 24991 & 4 \\ 71 & 404 \\ \underline{17} & \underline{8510} \\ 310696 & 6550 \end{array}$$

2. "Long" multiplications like the above rarely occur in practice. Generally we have to multiply two numbers known only approximately, to a certain number of significant figures. Suppose we want to find the product of 3.5702 and 8.7025, five significant figures only being known. It is then useless to calculate the figures to the right of the vertical line in the scheme above. To omit this useless part we proceed as follows. In multiplying by 8, place a dot over the last figure 2 of the multiplicand: in multiplying by 7, place a dot over the 0 of the multiplicand, beginning the multiplication with this figure (adding, however, the 1 which is to be carried from the preceding product 7 × 2): then to indicate the multiplication by 0 simply place a dot over the 7 of the multiplicand: the multiplication by 2 has then to begin at the 5 of the multiplicand. Thus we obtain:

$$\begin{array}{r|l}
\dot{3}.\dot{5}\dot{7}\dot{0}\dot{2} & 8.70\underline{2}5 \\
\hline
28\ 5616 & \\
2\ 4991 & \\
71 & \\
18 & \\
\hline
31.0696 &
\end{array}$$

The last figure so found is slightly uncertain, just as the last figures of the given numbers generally are.

3. In *division* it is most convenient to place the divisor to the right of the dividend. Thus

$$\begin{array}{r|l}
27.9823 & 3.1416 = 8.90702 \\
25\ 1328 & \\
\hline
2\ 8495 & 0 \\
2\ 8274 & 4 \\
\hline
220 & 600 \\
219 & 912 \\
\hline
 & 68800 \\
 & 62832
\end{array}$$

To cut off the superfluous part to the right of the vertical line, subtract the first partial product as usual; then cut off the last figure from the divisor and divide by the remaining portion; go on in this way, cutting off a figure from the divisor at every new division until the divisor is used up. Thus:

$$\begin{array}{r|l}
27.9823 & 3.1416 = 8.90701 \\
25\ 1328 & \\
\hline
2\ 8495 & \\
2\ 8274 & \\
\hline
221 & \\
220 & \\
\hline
1 &
\end{array}$$

ANSWERS

[Answers which might in any way lessen the value of the Exercise are not given.]

Pages 9–10. **5.** $2\frac{1}{4}$ miles. **16.** 173.9 ft.

Pages 13–14. **3.** 22. **4.** $\frac{1}{2}(bc + ca + ab)$.
7. $\frac{1}{2}(a^2 + 2\,bc - 2\,ca - b^2) = \frac{1}{2}(a - b)(a + b - 2\,c)$.

Pages 17–18. **4.** $\frac{1}{2}r_1r_2 \sin(\phi_2 - \phi_1)$.
5. $\frac{1}{2}[r_2r_3 \sin(\phi_3 - \phi_2) + r_3r_1 \sin(\phi_1 - \phi_3) + r_1r_2 \sin(\phi_2 - \phi_1)]$.
6. $\dfrac{2\,r_1r_2}{r_1 + r_2}\cos\frac{1}{2}(\phi_2 - \phi_1)$. **7.** $r\cos\phi = x + y\cos\omega,\ r\sin\phi = y\sin\omega$.

Page 22. **17.** They intersect at $[\frac{1}{4}(x_1+x_2+x_3+x_4),\ \frac{1}{4}(y_1+y_2+y_3+y_4)]$.
20. $[\frac{1}{3}(x_1 + x_2 + x_3),\ \frac{1}{3}(y_1 + y_2 + y_3)]$.

Page 35. **21.** $P = 1000(1 + r)$; $P = 1000 + 60\,n$.

Page 38. **14.** No.

Page 45. **1.** (*e*) $\sin^2\beta$; (*f*) $a_2a_3+a_3a_1+a_1a_2$.
8. (*b*) (4, 3), (4, − 3), (− 4, 3), (− 4, − 3); (*d*) (3, − 2);
(*e*) $(\pm\frac{1}{3}, \pm 3)$; (*f*) $(\frac{2}{3}, \frac{1}{3})$.

Pages 48–49. **1.** (*a*) 0; (*b*) 0; (*c*) − 113; (*d*) − 5; (*e*) 1.
4. (*a*) (2, − 1, 3); (*b*) (83/41, − 81/41, − 35/41); (*c*) (− 5, 3, − 2);
(*d*) (± 3, ± 2, ± 4); (*e*) $(\pm 1, \pm 1, \pm\frac{1}{2})$; (*f*) (1, 0, − 3).

Page 53. **1.** (*a*) 0; (*b*) − 180; (*c*) − 27846; (*d*) 7728; (*e*) 36;
(*f*) 550.

Page 57. **6.** (27/2, − 77/2).

Pages 59–60. **6.** 640/39. **9.** $(b_1m_2 - b_2m_1)^2/2\,m_1m_2(m_1 - m_2)$.
10. $(3, \frac{4}{3})$.

Pages 65–66. **2.** (*a*) $r\sin\phi = \pm 5$; (*b*) $r\cos\phi = \pm 4$;
(*c*) $r\cos(\phi - \frac{5}{6}\pi) = \pm 12$.
3. $\phi = 0,\ r\sin\phi = 9,\ \phi = \frac{1}{2}\pi,\ r\cos\phi = 6$. **14.** 8464/85.
19. (− 5, − 10). **21.** $x = 1$ (by inspection), $4\,x - 3\,y + 16 = 0$.

Page 68. **4.** $h^2 - ab \gtreqless 0.$

Page 69. **1.** $\tan^{-1}\dfrac{2\sqrt{h^2 - ab}}{a + b}$; $a = -b$. $h^2 = ab$.
4. $[m_1(b_2 - b) - m_2(b_1 - b)]^2/2\, m_1 m_2(m_2 - m_1)$.
6. $r(2 \cos \phi - 3 \sin \phi) + 12 = 0$.
10. 1 hr. 10 m.; 176 miles from Detroit.

Page 75. **6.** 560. **7.** 120. **8.** 55200. **9.** 60; 24, 36.
10. 487635, 32509, 1653.
11. ${}_nC_{\frac{1}{2}n}$, when n is even; ${}_nC_{\frac{1}{2}(n-1)} = {}_nC_{\frac{1}{2}(n+1)}$, when n is odd.
12. 66. **13.** 120.

Pages 82–83. **2.** $a_0x^3 + a_1x^2 + a_2x + a_3$. **4.** $8\, abcd$.
6. (*a*) $x = 2$, $y = -1$, $z = 2$, $w = 3$; (*b*) $x = 1$, $y = 3$, $z = 2$, $w = -1$.
7. (*a*) No; (*b*) Yes.
8. $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma + 2 \cos \alpha \cos \beta \cos \gamma = 1$.

Pages 85–86. **2.** (*a*) $ABC + 2\, FGH - AF^2 - BG^2 - CH^2$;
(*b*) $x^2 + y^2 + z^2 - 2(yz + zx + xy)$; (*c*) $-(x^2 + y^2 + z^2)$; (*e*) 4.
7. (*a*) $1 + a^2 + b^2 + c^2$; (*b*) $(ad + cf - be)^2$; (*c*) $(ad + be + cf)^2$.

Pages 90–91. **6.** $x^2 + y^2 - 96\, x - 54\, y + 2408 = 0$; 31.8 ft. or 66.3 ft.
8. $x^2 + y^2 - 16\, x + 8\, y + 60 = 0$. **9.** A circle except for $\kappa = \pm 1$.
10. $x^2 + y^2 + 4\,\dfrac{1 + k^2}{1 - k^2}\,x + 4 = 0$.

Page 92. **2.** (*a*) $r^2 - 20\, r \sin \phi + 75 = 0$;
(*b*) $r^2 - 12\, r \cos (\phi - \frac{1}{4}\pi) + 18 = 0$; (*c*) $r + 8 \sin \phi = 0$.

Page 94. **8.** $x^2 - 6\, x + 28 = 0$. **9.** $x^2 + 2\, pmx + qm^2 = 0$.

Page 96. **3.** $(-6, -1)$, $(29/106, 42/53)$.
7. $8\, x - 4\, y - 11 \pm 15\sqrt{2} = 0$.

Page 98. **3.** $(x_1 - h)(x - h) + (y_1 - k)(y - k) = r^2$.
7. $(-r^2A/C, -r^2B/C)$. **8.** $(2, 1)$.

Page 100. **6.** $(x - 79/38)^2 + (y - 55/38)^2 = (65/38)^2$.
8. $x^2 + y^2 + 4\, x - 2\, y - 15 = 0$.

Page 105. **1.** (*c*) Polar lies at infinity.

Pages 108–109. **3.** Let L, M be the intersections of the circle with CP_1, then $d^2 - r^2 = LP_1 \cdot MP_1$.

4. $x = y$; $\sqrt{\frac{1}{2}(a+b)^2 - 4c}$.

6. (*c*) $2x^2+2y^2+22x+6y+15=0$. $2x^2+2y^2-10x-10y-25=0$.

9. $bx^2 + by^2 + a^2mx - a^2y = 0$.

12. If the vertices of the square are (0. 0), $(a, 0)$, $(0, a)$, (a, a) and k^2 is the constant, the locus is $2x^2 + 2y^2 - 2ax - 2ay + 2a^2 - k^2 = 0$; $k > a$; $\frac{1}{2}a\sqrt{6}$.

13. If the vertices of the triangle are $(a. 0)$. $(-a. 0)$, $(0, a\sqrt{3})$ and k^2 is the constant, the locus is $3x^2 + 3y^2 - 2\sqrt{3}\,ay + 3a^2 - 2k^2 = 0$.

Page 126. **8.** (*a*) $(3 + 4i)/25$: (*b*) $(3 + \sqrt{5}i)/14$; (*c*) $(-5 + 3i)/34$; (*d*) $(1 - 6i)/37$.

Page 130. **7.** (*g*) $\pm\frac{1}{2}(\sqrt{6}+\sqrt{2}i)$; (*h*) $\sqrt[3]{2}(\cos 80^\circ + i\sin 80^\circ)$, $\sqrt[3]{2}(\cos 200^\circ + i\sin 200^\circ)$, $\sqrt[3]{2}(\cos 320^\circ + i\sin 320^\circ)$.

Pages 135–136. **10.** (*a*) $2y = 3x^2 + 5x$; (*b*) $12y = -5x^2 + 29x - 18$.

11. $300y = -x^2 + 230x$; 44.1 ft. above the ground; 230 ft. from the starting point.

20. (*b*) No parabola of the form $y = ax^2 + bx + c$ is possible.

Page 138. **13.** (2. 3), (– 1.8, 3.6), (3.1, – 2, 8), (– 3.3. – 3.8).

Page 142. **6.** East, East 33° 41′ North, East 53° 8′ North, East 18 26′ South.

10. $100/(\pi + 4)$.

Pages 145–146. **10.** 0, 8° 8′. **11.** 7° 29′.

15. When the side of the square is 3 in.

18. (*a*) $6y = x^3 + 6\,{}^2x - 19x$; (*b*) $7y = 2x^3 - x^2 - 29x + 35$.

Page 147. **1.** (*a*) –1, 3.62, 1.38; (*b*) –1.45. –.403, .855; (*c*) –1.94, .558, 1.38; (*d*) 2.79.

Page 154. **4.** (*d*) $-252x^{\frac{5}{2}}y^{\frac{5}{2}}$; (*g*) $40a^6b^4 - 80a^4b^6$; (*h*) $27/a^{25}$.

Page 159. **3.** (*a*) $p_1p_2=p_3$; (*b*) $p_1{}^3p_3=p_2{}^3$; (*c*) $p_1{}^6=27p_2{}^3=729p_3{}^2$.

Page 162. **1.** – 1.88. 1.53, .347.

Page 167. **1.** (*a*) 4.06155; (*b*) ± 2.08779; (*c*) 1.475773.

2. 2.0945514. **3.** .34899.

4. (*a*) (1.88, 3), (– 1.53, 3), (– .347, 3); (*b*) (.309, 1.10), (1.65, 1.55), (–1.96. .347); (*c*) (–2.106. –1.0265).

5. 3.39487 in. **6.** 9.69579 ft. **7.** – 2, 1 ± $\sqrt{3}$.

8. .22775, 3.1006. **9.** 5.4418 ft.

10. (2. 3), (– 1.848, 3.584), (3.131, – 2.805), (– 3.383, – 3.779).

11. (2.21. .89). **12.** .34729 a.

Pages 173–174. **2.** (*a*) $(4, \frac{1}{3}\pi)$, $(4, \frac{5}{3}\pi)$; (*b*) $(a, \frac{1}{2}\pi)$, $(a, \frac{3}{2}\pi)$; (*c*) $(4, 0)$; (*d*) $(4a, \frac{1}{3}\pi)$, $(4a, \frac{5}{3}\pi)$.
7. (*a*) $y^2 - 4x + 4 = 0$; (*b*) $14y^2 - 45x + 52y + 60 = 0$.
8. (*b*) $x^2 - 10x - 3y + 21 = 0$; (*c*) $x^2 + 2x + y - 1 = 0$.
9. The equation of a parabola contains an xy term when its axis is oblique to a coordinate axis.

Pages 179–180. **1.** (*a*) $18x - 30$;
(*b*) $6x^5 - 30x^4 + 48x^3 - 24x^2 + 8x - 8$.
2. (*a*) $y' = 5/2y$; (*b*) $y' = 6/(5 - 2y)$; (*c*) $y' = 2/3y$.
5. (*a*) $y' = -y/x$; (*b*) $y' = (6 - 2xy)/x^2$;
(*c*) $y' = -(Ax + Hy + G)/(Hx + By + F)$.

Pages 186–188. **8.** (*a*) $y = 0$; (*b*) $2x + 2y - 9 = 0$, $2x - y - 18 = 0$;
(*c*) $2x + 2y - 9 = 0$, $8x + 16y - 27 = 0$, $24x - 16y - 153 = 0$;
(*d*) $8x - 16y - 27 = 0$.
14. $y = kx$. **15.** Directrix; $y^2 = a(x - 3a)$. **22.** $\frac{4a}{m^2}(1 + m^2)$
29. $x^2 - 80x - 2400y = 0$; $0, -\frac{1}{2}, -\frac{2}{3}, -\frac{1}{2}, 0, \frac{5}{6}, 2$.
30. $x^2 = 360(y - 20)$.

Pages 194–195. **2.** $(3\pi - 4)/6\pi$. **3.** $\frac{8}{3}a^2\frac{(1+m^2)^{\frac{3}{2}}}{m^3}$.
8. (*a*) $64/3$; (*b*) $625/12$; (*c*) $1/12$. **9.** 123.84 ft^3. **10.** $1794\frac{1}{3}$ tons.
11. 199.4 ft^2.

Page 197. To obtain the following solutions, take the origin at one end of the beam and the axis Ox along the beam.
1. $F = W$, $M = W(x - l)$. **2.** $F = w(\frac{1}{2}l - x)$, $M = \frac{1}{2}w(l - x)x$.
3. (*a*) $F_1 = -wx$, $M_1 = -\frac{1}{2}wx^2$; $F_2 = w(\frac{1}{2}l - x)$, $M_2 = -\frac{1}{2}w(\frac{1}{3}l^2 - lx + x^2)$;
$F_3 = w(l - x)$, $M_3 = -\frac{1}{2}w(l - x)^2$;
(*b*) $F_1 = -W$, $M_1 = -Wx$; $F_2 = 0$; $M_2 = -\frac{1}{3}Wl$;
$F_3 = W$, $M_3 = -W(l - x)$.
4. (*a*) $F_1 = \frac{1}{6}wl$, $M_1 = \frac{1}{6}wlx$; $F_2 = w(\frac{1}{2}l - x)$,
$M_2 = -\frac{1}{2}w(x^2 - lx + \frac{1}{9}l^2)$; $F_3 = -\frac{1}{6}wl$, $M_3 = \frac{1}{6}wl(l - x)$.

Page 200. **9.** $8x^2 - 2xy + 8y^2 - 63 = 0$.

Page 204. **10.** $3x^2 - y^2 = 3a^2$. **11.** b. **14.** $2xy = 1$.

Pages 211–212. **2.** $\frac{a^2}{x}X + \frac{b^2}{y}Y = c^2$. **13.** 54.5 ft., 42.2 ft. **18.** b^2/a^2.
23. An ellipse or hyperbola according as one circle lies within or without the other circle.

Pages 221–222. **7.** (*a*) $A^2a^2 - B^2b^2 = C^2$;
(*b*) $a^2\cos^2\beta - b^2\sin^2\beta = p^2$.
19. b^2. **21.** $a^2 + b^2$; $a^2 - b^2$.
22. $4\,ab$. **23.** $\sin^{-1}(ab/a'b')$.
25. (*a*) $x^2 + y^2 = a^2 + b^2$; (*b*) $x^2 + y^2 = a^2 - b^2$.

Page 227. **3.** (*a*) $(1, -1)$, $(1 \pm \sqrt{2}, -1)$, $x = 1 \pm \frac{3}{2}\sqrt{2}$;
(*b*) $(\frac{1}{2}, 0)$, $(\frac{5}{2}, 0)$, $(-\frac{3}{2}, 0)$, $x = 0$, $x = 1$.
4. $2\,b^2/a$. **8.** (*a*) $a^2y^2 = b^2x(a - x)$; (*b*) $b^2x^2 = a^2y(b - y)$.
10. Two straight lines.

Page 235. **2.** (*a*) Vertices $(5, 3)$, $(8, 3)$; semi-axes $3/2$, $\sqrt{2}$.
(*b*) Vertices $(4, 8/3)$, $(8, 8)$; semi-axes $10/3$, $5\sqrt{3}/3$.
(*c*) vertices $(17/5, 7/5)$, $(1, 3)$: semi-axes $\sqrt{65}/5$, $\sqrt{13}/2$.
3. $3\,x + 2\,y - 2 = 0$; $(21/13, -37/26)$, $10/\sqrt{13}$.

Page 237. **5.** $(a\cos\theta, -a\sin\theta)$, $x^2 + y^2 - 2\,a(x\cos\theta - y\sin\theta) = 0$.

Pages 246–247. **2.** (*a*) $3\,x - 14\,y = 0$; (*b*) $y = -3/13$, $x = -14/13$.
5. $2\,x^2 - xy - 15\,y^2 + x + 19\,y - 6 = 0$,
$2\,x^2 - xy - 15\,y^2 + x + 19\,y - 28 = 0$.
6. $6\,x^2 + xy - 2\,y^2 - 9\,x + 8\,y - 46 = 0$,
$6\,x^2 + xy - 2\,y^2 - 9\,x + 8\,y + 34 = 0$.
11. (*a*) $x^2/4 + y^2 = 1$; (*b*) $x^2/4 - y^2/2 = 1$; (*c*) $3\,x^2 + y^2 + 6 = 0$;
(*d*) $x^2/16 + y^2/4 = 1$; (*e*) $(3 + \sqrt{17})x^2 + (3 - \sqrt{17})y^2 = 4$;
(*f*) $(2 + \sqrt{2})x^2 + (2 - \sqrt{2})y^2 = 1$.
15. $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$.
19. Equilateral hyperbola.

Page 253. **2.** (*a*) Simple point; (*b*) node; (*c*) cusp; (*d*) cusp.
4. (*a*) None; (*b*) node at $(b, 0)$; (*c*) isolated point at $(a, 0)$;
(*d*) cusp at $(a, 0)$.

Page 260. **4.** $r = a(\sec\phi \pm \tan\phi)$ or $(x - a)y^2 + x^2(x + a) = 0$.
10. $x^2y^2 = a^2(x^2 + y^2)$. **11.** Cissoid $(a - x)y^2 = x^3$.
12. $y(x^2 + y^2) = a(x^2 - y^2)$. **13.** $r = a \operatorname{ctn}\phi$.
14. $(x^2 + y^2)^2 = 4\,ax(x^2 - y^2)$.

Page 283. **6.** $\dfrac{l + l'}{\sqrt{2(1 + ll' + mm' + nn')}}$, etc.
13. $\frac{1}{3}(x_1 + x_2 + x_3)$, $\frac{1}{3}(y_1 + y_2 + y_3)$, $\frac{1}{3}(z_1 + z_2 + z_3)$.

Page 287. **6.** $\cos^{-1}(7/3\sqrt{29})$.

Page 291. **2.** $\frac{1}{2}\sqrt{465}$. **3.** $\frac{5}{2}\sqrt{269}$.

6. $(3962, 47° 43', 276° 16')$, $(320, -2914, 2666)$, 2931.

7. $\frac{1}{2} r_1 r_2 \sqrt{1 - [\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2 \cos(\phi_1 - \phi_2)]^2}$.

8. $\sqrt{r_1^2 + r_2^2 - 2\, r_1 r_2 [\sin\theta_1 \sin\theta_2 \cos(\phi_1 - \phi_2) + \cos\theta_1 \cos\theta_2]}$.

10. $-1, 10, 7$.

Page 296. **3.** $39\,x - 10\,y + 7\,z - 89 = 0$.

5. $97/28, -97/49, -97/9$. **7.** $3\,x - 4\,y + 2\,z - 6 = 0$.

Page 300. **5.** $4\,x + 8\,y + z = 81$, $4\,x + 8\,x + z = 90$.

Page 303. **2.** (*a*) $56/3$; (*b*) 0; (*c*) $19/3$.

Page 306. **12.** $3\,x - 2\,y = 1$. **13.** $6\,x + 11\,y + 9\,z = 58$.

16. $70° 31'$. **17.** $\cos^{-1}(2\,h^2 + 3\,a^2)/(4\,h^2 + 3\,a^2)$.

Pages 314–316. **3.** $69° 29'$. **19.** (*a*) $\sqrt{63/19}$; (*b*) $\sqrt{194/33}$.

21. $x - 2\,y + z + 8 = 0$.

24. $$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0.$$

Page 320. **11.** $(-3, -3, 2)$, $(9, 9, -6)$.

Pages 325–326. **4.** $(1, 0, -3)$, $(-9/11, 20/11, 27/11)$.

7. $x^2 - 3\,y^2 - 3\,z^2 = 0$. **13.** $25(x^2 + y^2 + z^2) = 16^2$, $25\,z = 64$.

Pages 329–331. **4.** $(4, -5, -3)$. **5.** $(4, 6, 2)$.

6. $5\,x + 2\,y - z = 25$, $2\,x - 3y + z + 25 = 0$.

20. $9\,x^2 + 4\,y^2 + 13\,z^2 + 2\,xy - 273 = 0$.

21. $(x - lk)^2 + (y - mk)^2 + (z - nk)^2 = r^2$.

22. $[l(x+h) + m(y+j) + n(z+k)]^2 - [(x+h)^2 + (y+j)^2 + (z+k)^2 - r^2] = 0$.

Page 336. **3.** $\sqrt{a^2 - c^2}\, x \pm \sqrt{b^2 - c^2}\, z = 0$.

5. $(x^2 + y^2 + z^2 - a^2 - b^2)^2 - 4\,b^2(a^2 - y^2) = 0$.

8. (*a*) $16\,a^2(x^2 + z^2) = y^4$; (*b*) $16\,a^2[(x + a)^2 + z^2] = (4\,a^2 - y^2)^2$.

9. $y^2(x^2 + z^2) = a^4$.

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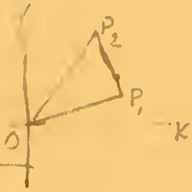
$x = x_1 + k_1 \cdot \Delta x$

$r = \sqrt{x^2 + y^2}$

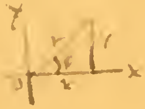
$d = \sqrt{(\Delta x)^2 + (\Delta y)^2}$

$A = \frac{1}{2}\begin{vmatrix} x_1 y_1 \\ x_2 y_2 \end{vmatrix} = \frac{1}{2}(x_1 y_2 - x_2 y_1)$



$x = r\cos\phi \qquad r = \sqrt{x^2 + y^2}$

$y = r\sin\phi \qquad \tan\phi = \frac{y}{x}$



Straight line

$y = mx.$

$y = mx + b$

$A = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

Lines are parallel if $m_1 = m_2$

" " perpendicular

if $m_1 m_2 = -1$

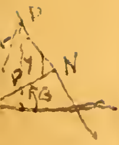
$\frac{x}{a} + \frac{y}{b} = 1$

$a = -\frac{C}{A}, \ b = -\frac{C}{B}$

$\tan\theta = \tan(a' - a)$

$= \frac{\tan a' - \tan a}{1 + \tan a \tan a'}$

$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1$



$x\cos B + y\sin B = P$

$(x-h)^2 + (y-k)^2 = r^2$

$r^2 + r_1^2 - 2rr_1\cos(\phi - \phi_1) = a^2$

$r = 2a\cos\phi$